MATURA ROZSZERZONA 2019 r.

Zad. 1 <1p>   dla $x, y > 0$ i $x, y \neq 1$

$$(\log_{\frac{1}{x}} y)\cdot(\log_{\frac{1}{y}} x) = \frac{\log y}{\log \frac{1}{x}} \cdot \frac{\log x}{\log \frac{1}{y}} = \frac{\log x \cdot \log y}{\log x^{-1} \cdot \log y^{-1}} =$$

$$= \frac{\log x \cdot \log y}{-\log x \cdot (-\log y)} = \frac{\log x \cdot \log y}{\log x \cdot \log y} = 1$$

D

Zad. 2 <1p>

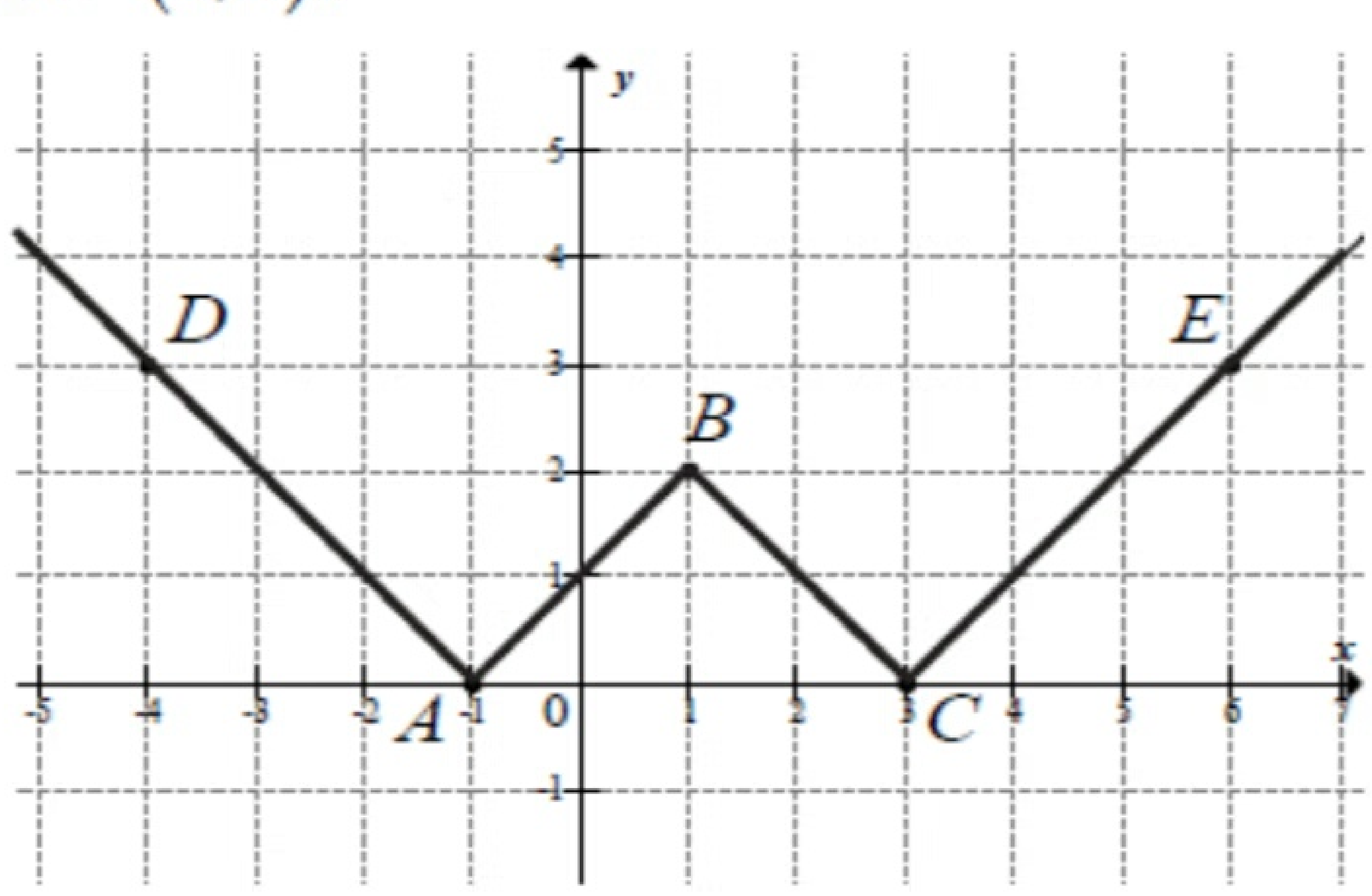
$$\cos^2 105^\circ - \sin^2 105^\circ = \cos 210^\circ = \cos(180^\circ + 30^\circ) =$$

$$= -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

A

Zadanie 3. (0–1)

Na rysunku przedstawiono fragment wykresu funkcji $y = f(x)$, który jest złożony z dwóch półprostych AD i CE oraz dwóch odcinków AB i BC, gdzie $A = (-1, 0)$, $B = (1, 2)$, $C = (3, 0)$, $D = (-4, 3)$, $E = (6, 3)$.



Wzór funkcji f to

A. $f(x) = |x+1| + |x-1|$

B. $f(x) = ||x-1| - 2|$

C. $f(x) = ||x-1| + 2|$

D. $f(x) = |x-1| + 2$

Np.:

Przekształce-niami

$f_0(x) = |x-2|$

$\downarrow |x|$

$f_0(|x|) = ||x|-2|$

$\downarrow [1;0]$

$f(x) = f_0(|x-1|) = ||x-1|-2|$

B

Zad. 4 <1p>

$A, B \subset \Omega$

(1) $P(B') = \frac{1}{4}$

(2) $P(A|B) = \frac{1}{5}$

$P(A \cap B) = ?$

(1) $P(B) = 1 - P(B') = 1 - \frac{1}{4} = \frac{3}{4}$

(2) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$P(A \cap B) = P(B) \cdot P(A|B) = \frac{3}{4} \cdot \frac{1}{5} = \frac{3}{20}$$

C

Zad. 5 <2p>

$$\lim_{n \to \infty} \left(\frac{9n^3 + 11n^2}{7n^3 + 5n^2 + 3n + 1} - \frac{n^2}{3n^2 + 1} \right) = \frac{9}{7} - \frac{1}{3} =$$

$$= \frac{27 - 7}{21} = \frac{20}{21} \approx 0{,}952381$$

Odp. 9 5 2

Zad. 6 <3p>

$Z = \{1, 3, 5, 7, 9\}$
$n = 5$, $k = 5$
KI – kolejność istotna
BP – bez powtórzeń
Ω – liczby 5-cyfrowe

X – suma liczb z Ω ?

(1) $\Omega = \{97531; 97513; 97351; 97315; \dots$
$\dots 13579, 31579, 15379; 51379\}$

(2) $\overline{\overline{\Omega}} = 5! = 120$ – ILOŚĆ WSZYSTKICH TAKICH LICZB

(3) Liczba 5-cio CYFROWA MA WZÓR

$$L = 10000 \cdot a + 1000 \cdot b + 100 \cdot c + 10 d + e$$

np: $a = 1$; $b, c, d, e \in \{2, 3, 4, 5\}$

$$\underbrace{\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}} = 24$$

Przykładowo $a = 1$ występuje 24 razy
Analogicznie każda z cyfr $\{1, 3, 5, 7, 9\}$ występuje w liczbie L na każdym miejscu 24 razy oraz $\underbrace{1+3+5+7+9}_{25}$
Stąd po dodaniu otrzymamy:

$$X = 24(10000 \cdot 25 + 1000 \cdot 25 + 100 \cdot 25 + 10 \cdot 25 + 25)$$
$$= 24 \cdot 25 \cdot (10000 + 1000 + 100 + 10 + 1)$$
$$= 24 \cdot 25 \cdot 11111 = \underline{\underline{6666600}}$$

odp.

Zad. 7 <2p>

$f: \ y = 2x^2 + x + 2219$

$h: \ y = ax + b = f'(x_0)(x - x_0) + f(x_0)$ $\qquad b = ?$ ✓

$P = (\underbrace{10}_{x_0}; \underbrace{2429}_{y_0}) \in (f \cap h)$

(1) $f'(x) = 4x + 1 \longrightarrow f'(x_0 = 10) = 41 \ \wedge \ f(10) = 2429$

(2) $b = f(x_0) - f'(x_0) \cdot x_0$

$b = 2429 - 41 \cdot 10 = 2429 - 410 = 2019$

Odp.

Zad. 8 <3p>

Z: $\{a, x, y\} \in \mathbb{R}^+$, $x < y$

T: $\dfrac{x+a}{y+a} + \dfrac{y}{x} > 2$

D: (*) $\dfrac{x+a}{y+a} + \dfrac{y}{x} > 2 \quad / \cdot x(y+a) > 0$

$x(x+a) + y(y+a) > 2x(y+a)$

$x^2 + ax + y^2 + ay > 2xy + 2ax$

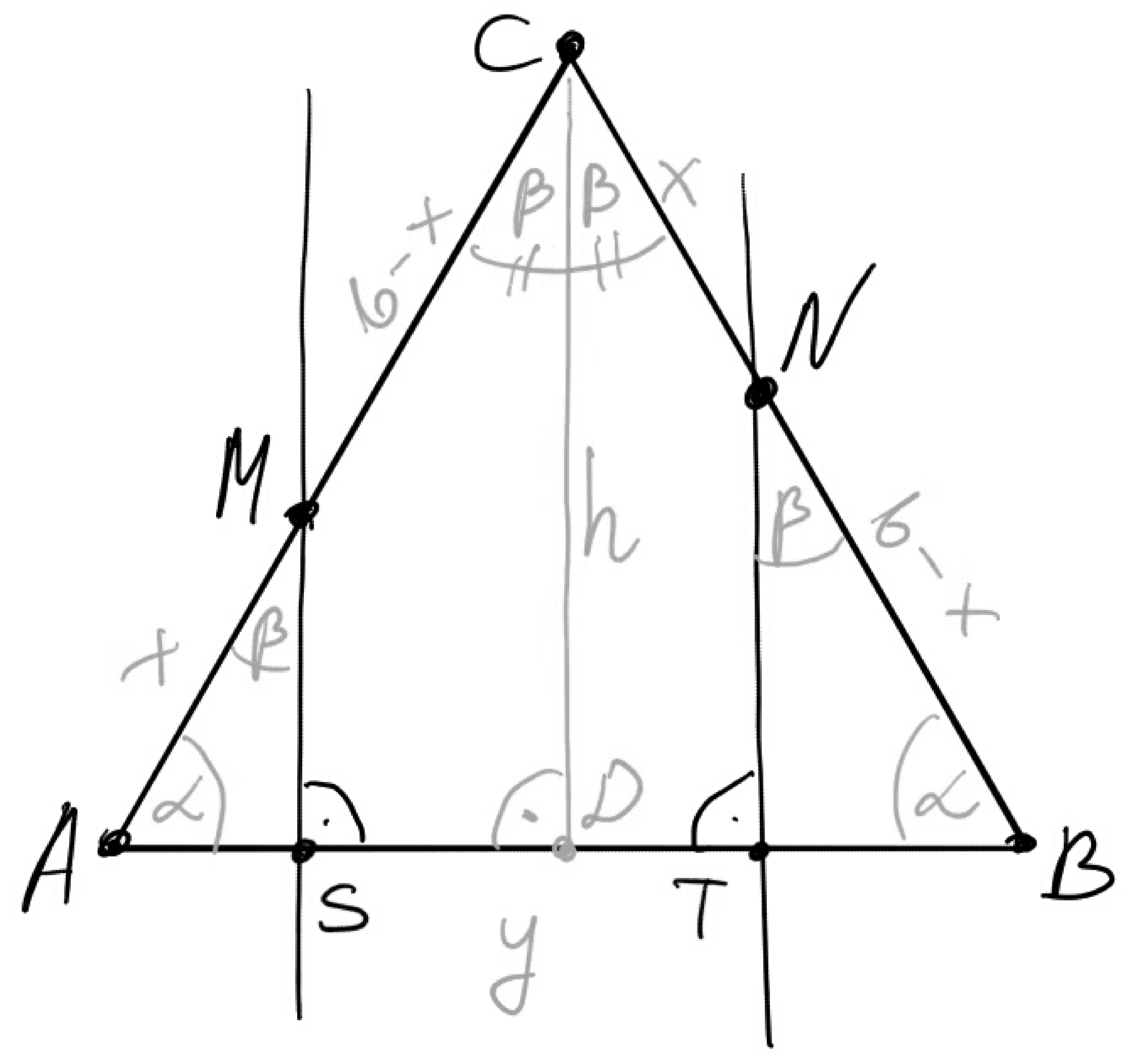
$x^2 - 2xy + y^2 + ax + ay - 2ax > 0$

$(x-y)^2 + ay - ax > 0$

$\underbrace{\underbrace{(x-y)^2}_{>0} + \underbrace{\underbrace{a}_{>0} \cdot \underbrace{(y-x)}_{>0}}_{>0}}_{>0} > 0 \quad \wedge \quad x < y \Downarrow y - x > 0$

cnd.

Zad. 9 <3p>



Z: $|AB| = a$

$|BC| = |AC| = b$

$|AM| = |NC| = x$

$|ST| = y$

$|CD| = h$

T: $|ST| = \frac{1}{2}|AB|$

$$\Updownarrow$$

$$y = \frac{1}{2}a$$

D: (1) $\overline{MS} \parallel \overline{NT} \parallel \overline{CD} \perp \overline{AB} \ \wedge \ |AD| = |DB| = \frac{1}{2}a$

(2) $\triangle ASM \overset{kkk}{\sim} \triangle BTN \overset{kkk}{\sim} \triangle ADC \quad \underbrace{(\overset{k}{\alpha}, \overset{k}{90^\circ}, \overset{k}{\beta})}$

(3) $\frac{x}{|AS|} = \frac{b-x}{|SD|} = \frac{x}{|DT|} = \frac{b-x}{|TB|} \Rightarrow \begin{cases} |AS| = |DT| \\ |SD| = |TB| \end{cases}$

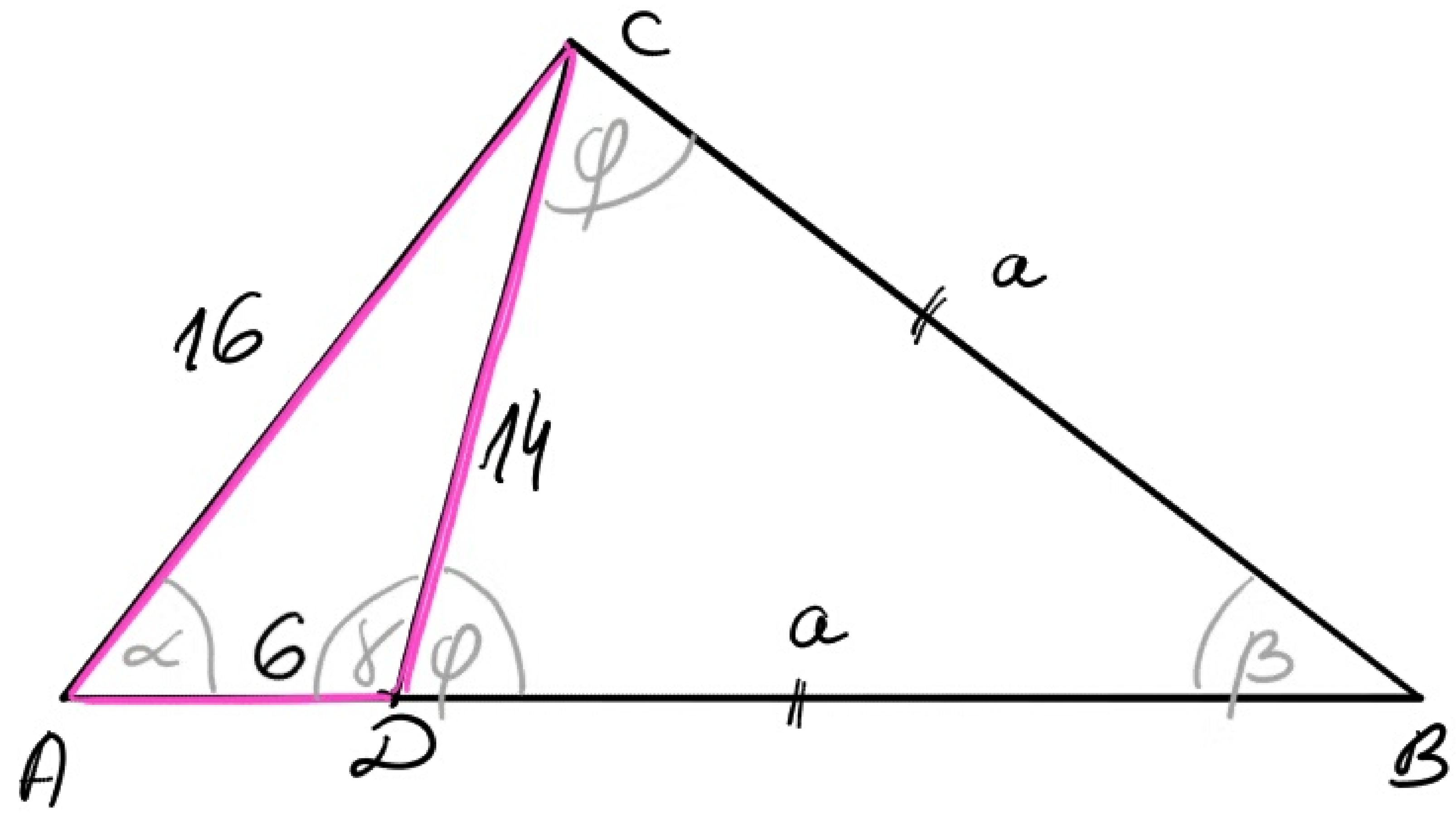
$$|AS| + |SD| = \frac{1}{2}a \quad \wedge \quad |AS| = |DT|$$

$$|DT| + |SD| = \frac{1}{2}a \quad \wedge \quad |DT| + |SD| = y$$

$$\underbrace{y = \frac{1}{2}a \Rightarrow |ST| = \frac{1}{2}|AB|}$$

cnd.

Zad. 10 <4p>



$Ob_{ABC} = Ob_{\Delta} = ?$

(1) ΔADC – Tw. cosin.: $16^2 = 6^2 + 14^2 - 2\cdot 6\cdot 14\cdot \cos\gamma$

$24 = -168\cdot\cos\gamma \quad /:(-168)$

$\underbrace{\cos\gamma = -\frac{1}{7}} \Rightarrow \gamma \in (90°; 180°)$

$\Downarrow$

$\underbrace{\varphi = 180° - \gamma}$

(2) ΔDBC – Tw. cosin.:

$\cancel{a^2} = 14^2 + \cancel{a^2} - 2\cdot a\cdot 14\cos(180° - \gamma) \quad /:14$

$0 = 14 + 2a\cos\gamma \quad /:2 \quad \wedge \quad \cos\gamma = -\frac{1}{7}$

$0 = 7 + a\left(-\frac{1}{7}\right) \quad /\cdot 7$

$0 = 49 - a$

$a = 49$

(3) $Ob_{\Delta} = 16 + 6 + 2\cdot 49 = 120$ [j]

Zad. 11 < 6p >

(1) $O_1: x^2+y^2-12x-8y+43=0$

(2) $O_2: x^2+y^2-2ax+4y+a^2-77=0$

(3) $(O_1 \cap O_2)$ – 1 pkt. wspólny $\Rightarrow$ są styczne

$a = ?$

(1) $O_1: (x-6)^2+(y-4)^2 = \overbrace{-43+36+16}^{9} \Rightarrow \begin{cases} S_1=(6;4) \\ r_1=3 \end{cases}$

(2) $O_2: (x-a)^2+(y+2)^2 = \underbrace{77-\cancel{a^2}+\cancel{a^2}+4}_{81} \Rightarrow \begin{cases} S_2=(a;-2) \\ r_2=9 \end{cases}$

$\overrightarrow{S_1S_2} = [a-6; -6] \Rightarrow |S_1S_2| = \sqrt{(a-6)^2+(-6)^2}$

(3) ROZPATRUJEMY DWA PRZYPADKI STYCZNOŚCI:

I STYCZNE ZEWNĘTRZNIE $\vee$ II STYCZNE WEWNĘTRZNIE

$|S_1S_2| = r_1+r_2 \quad \vee \quad |S_1S_2| = |r_2-r_1|$

$\sqrt{(a-6)^2+36} = 12 \quad /^2 \quad \vee \quad \sqrt{(a-6)^2+36} = 6 \quad /^2$

$(a-6)^2+36 = 144 \quad \vee \quad (a-6)^2+36 = 36$

$(a-6)^2 = 108 \quad /\sqrt{\ } \quad \vee \quad (a-6)^2 = 0$

$|a-6| = 6\sqrt{3}$

$a-6=0$

$a = 6 \mp 6\sqrt{3}$

$a = 6$

Odp: $a = \{6-6\sqrt{3}; 6; 6+6\sqrt{3}\}$

Zad. 12 <6p>

(1) $(a; b; c) \Rightarrow$ c. arytmet.

$a, b, c > 0$

(2) $\left(\frac{1}{a}; \frac{2}{3b}; \frac{1}{2a+2b+c}\right) \Rightarrow b_n = b_1 \cdot q^{n-1}$ | $q = ?$

(1) $2b = a + c \Rightarrow \underline{c = 2b - a}$

(2) $q = \frac{1}{2a+2b+c} \cdot \frac{3b}{2} = \frac{2}{3b} \cdot \frac{a}{1} \wedge \underline{c = 2b - a}$

$q = \frac{3b}{(2a+2b+2b-a)\cdot 2} = \frac{2a}{3b}$

$$\underbrace{q = \frac{3b}{2\cdot(a+4b)} = \frac{2a}{3b}}$$

$9b^2 = 4a^2 + 16ab$

$9b^2 = 4a(a+b)$ $\Big/ \cdot \frac{1}{2(a+b)\cdot 3b}$

$\frac{3b}{2(a+b)} = \frac{2a}{3b}$

$4a(a+4b) = 9b^2 \wedge q = \frac{2a}{3b}$

$4a^2 + 16ab - 9b^2 = 0$ $/ : (9b^2)$

$\frac{4a^2}{9b^2} + \frac{16a}{9b} - 1 = 0$

$\left(\frac{2a}{3b}\right)^2 + \frac{8}{3}\cdot\frac{2a}{3b} - 1 = 0 \wedge q = \frac{2a}{3b} > 0$

$q^2 + \frac{8}{3}q - 1 = 0$ bo $\{a, b, c\} > 0$

$\left(q + \frac{4}{3}\right)^2 - \frac{16}{9} - 1 = 0$

$\left(q + \frac{4}{3}\right)^2 = \frac{25}{9}$ $/\sqrt{\ }$

$\left|q + \frac{4}{3}\right| = \frac{5}{3} \Rightarrow \left(q = -\frac{4}{3} + \frac{5}{3} \vee q = -\frac{4}{3} - \frac{5}{3}\right)$

dla $q > 0$: $\Leftarrow \left(q = \frac{1}{3} \vee q = -3\right)$

Odp: $\underline{\underline{q = \frac{1}{3}}}$

Zad. 13 <6p>

$$W(x) = 2x^3 + (m^3+2)x^2 - 11x - 2(2m+1)$$

$m = ?$

$x = ?$

(1) $\begin{cases} W(2) = 0 \\ W(-1) = 6 \end{cases}$

(2) $W(x) \leq 0$

(1) $\begin{cases} 16 + 4(m^3+2) - 22 - 2(2m+1) = 0 \\ -2 + (m^3+2) + 11 - 2(2m+1) = 6 \end{cases}$

$$\begin{cases} 4(m^3+2) - 2(2m+1) = 6 \\ (m^3+2) - 2(2m+1) = -3 \end{cases} \quad -$$

$3(m^3+2) = 9 \quad /:3$

$m^3 + 2 = 3$

$m^3 = 1 \quad /\sqrt[3]{\ }$

$\boxed{m = 1} \Rightarrow W(x) = 2x^3 + 3x^2 - 11x - 6$

(2) $2x^3 + 3x^2 - 11x - 6 \leq 0$

	x^3			
	2	3	−11	−6
$x=2$	2	7	3	0

$(x-2)$

$(x-2)(2x^2+7x+3) \leq 0$

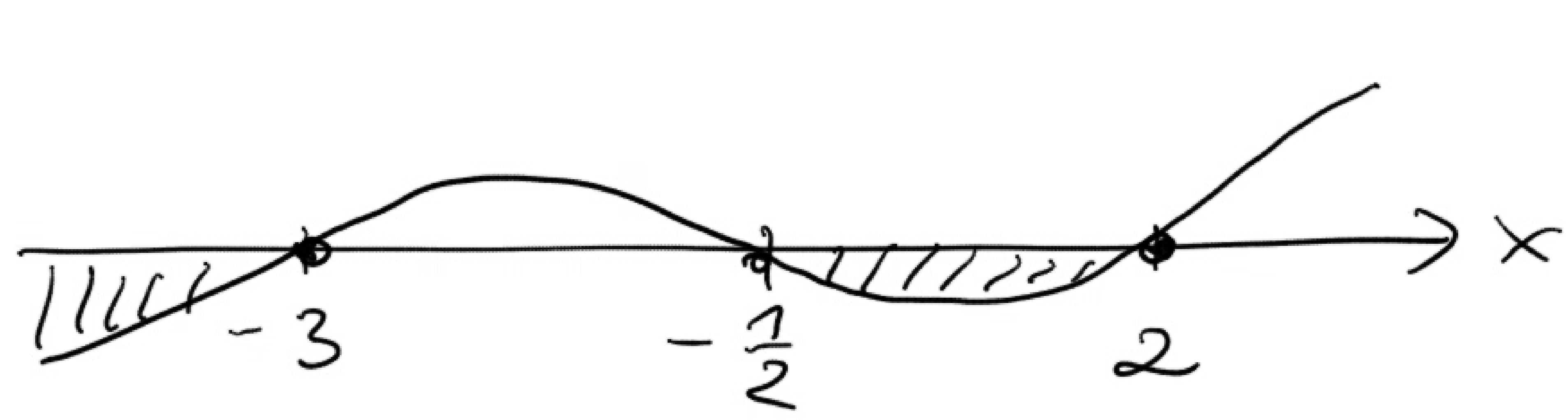
$2x^2 + 7x + 3 = 0$

$\Delta = 49 - 4 \cdot 2 \cdot 3 = 25 = 5^2$

$x_1 = \frac{-7-5}{2 \cdot 2} = \frac{-12}{4} = -3$

$x_2 = \frac{-7+5}{2 \cdot 2} = \frac{-2}{4} = -\frac{1}{2}$

$2(x-2)(x+3)(x+\frac{1}{2}) \leq 0$



Odp.: $m = 1$; $x \in (-\infty; -3\rangle \cup \langle -\frac{1}{2}; 2\rangle$

Zad. 14 <4p>

$$\cos x \cdot \left[\sin\left(x-\frac{\pi}{3}\right)+\sin\left(x+\frac{\pi}{3}\right)\right]=\frac{1}{2}\cdot\sin x$$

$$\cos x \cdot 2\sin x \cdot \cos\left(-\frac{\pi}{3}\right)=\frac{1}{2}\cdot\sin x$$

$$2\sin x\cos x \cdot \underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}}-\frac{1}{2}\sin x=0$$

$$\sin x\cdot\cos x-\frac{1}{2}\sin x=0$$

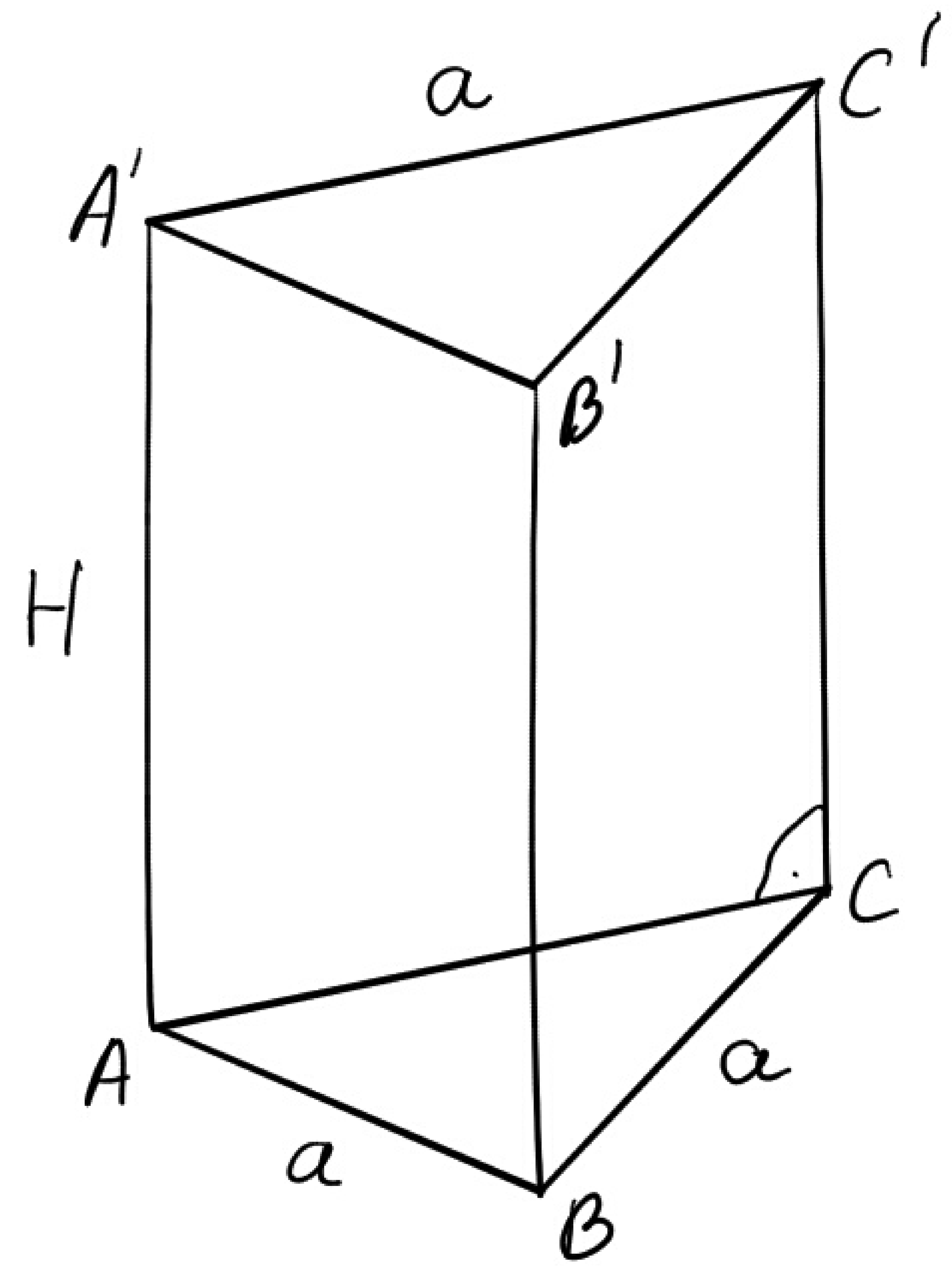
$$\sin x\left(\cos x-\frac{1}{2}\right)=0 \quad \wedge \quad k\in C$$

$$\sin x=0 \quad \vee \quad \cos x=\frac{1}{2}$$

$$x_1=k\pi \quad \vee \quad x_2=\frac{\pi}{3}+2k\pi \quad \vee \quad x_3=-\frac{\pi}{3}+2k\pi$$

Odp: $x=\left\{-\frac{\pi}{3}+2k\pi;\ k\pi;\ \frac{\pi}{3}+2k\pi\right\} \wedge k\in C$

Zad. 15 <7p.>



(1) $V = 2$

(2) $P_C = \min.$

$a = ?$
$H = ?$
$P_c = ?$

Zał. $a, H > 0$

(1) $\frac{a^2\sqrt{3}}{4} \cdot H = 2 \quad / \cdot \frac{4}{a^2\sqrt{3}}$

$\underbrace{H = \frac{8}{a^2\sqrt{3}}} > 0$

(2) $P_c = \cancel{2} \cdot \frac{a^2\sqrt{3}}{\cancel{4}_2} + 3 \cdot aH$

$P(a) = \frac{a^2\sqrt{3}}{2} + \cancel{3}^{\sqrt{3}}\cancel{a} \cdot \frac{8}{a^{\cancel{2}}\cancel{\sqrt{3}}}$

$P(a) = \frac{a^2\sqrt{3}}{2} + \frac{8\sqrt{3}}{a} = \frac{\sqrt{3}}{2}\left(a^2 + \frac{16}{a}\right) \quad \wedge \ \overbrace{a > 0}^{D:}$

(3) $P'(a) = \frac{\sqrt{3}}{2}\left(2a - \frac{16}{a^2}\right) = \sqrt{3}\left(a - \frac{8}{a^2}\right) = \sqrt{3} \cdot \frac{a^3 - 8}{a^2}$

$P'(a) = \frac{\sqrt{3}(a-2)(a^2+2a+4)}{a^2} \quad \wedge \ D' = D$

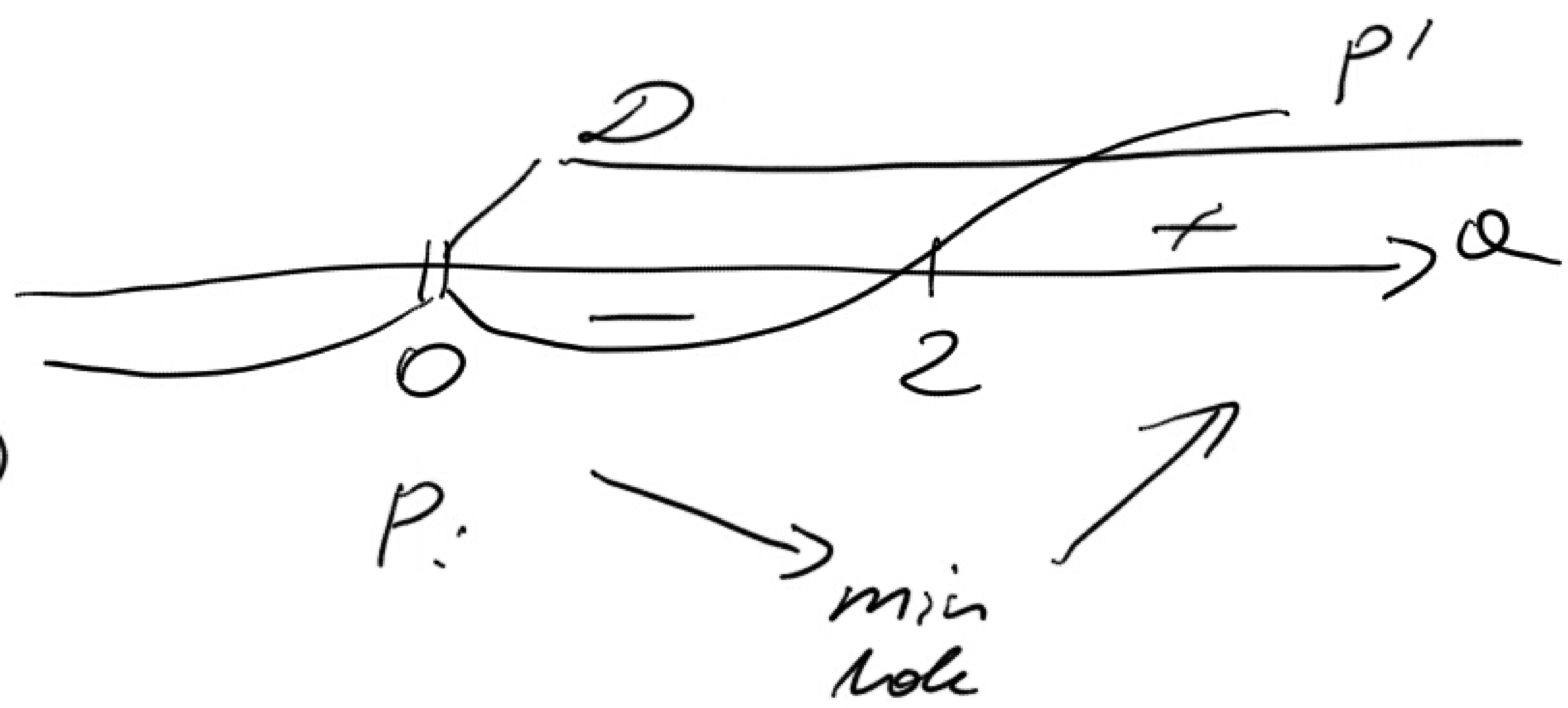
(4) $P'(a) = 0$ dla $a = 2$

$P'(a) < 0$ dla $a \in (0; 2)$

$P'(a) > 0$ dla $a \in (2; \infty)$

$\Downarrow$

$P = \max$ dla $\underbrace{a = 2}$



$\underbrace{H} = \frac{8}{4\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \underbrace{\frac{2\sqrt{3}}{3}}$

$P_c = \frac{\sqrt{3}}{2} \cdot \left(4 + \frac{16}{2}\right) = \frac{\sqrt{3}}{\cancel{2}} \cdot \cancel{12}^{6} = \underline{\underline{6\sqrt{3}}}$

Odp. $a = 2; \ H = \frac{2\sqrt{3}}{3}; \ P_c = 6\sqrt{3}$