Matura rozszerzona 9 maj 2018

Zad. 1.

$a=\frac{\sqrt[4]{8}}{2}$; $b=\frac{1}{2\cdot\sqrt[4]{8}}$; $c=\sqrt[4]{8}$; $d=\frac{2}{\sqrt[4]{8}}$; $k=2^{-3/4}$

$a=\frac{2^{3/4}}{2}=2^{-\frac{1}{4}} \Rightarrow \underline{\underline{a=k}}$ (A)

Zad. 2

$||x|-2|=|x|+2$

$|x|-2=|x|+2 \quad \vee \quad |x|-2=-|x|-2$

$-2=2 \qquad 2|x|=0 \quad /:2$

sprzeczne $\qquad |x|=0$

$x=\emptyset \qquad \boxed{x=0}$

(B)

1 rozwiązanie: $x=0$

Zad. 3

$$2\cdot\log_5 10-\frac{1}{\log_{20}5}=\log_5 10^2-\frac{1}{\frac{\log_5 5}{\log_5 20}}=$$

$$=\log_5 100-\log_5 20=\log_5\frac{100}{20}=\log_5 5=\underline{\underline{1}}$$

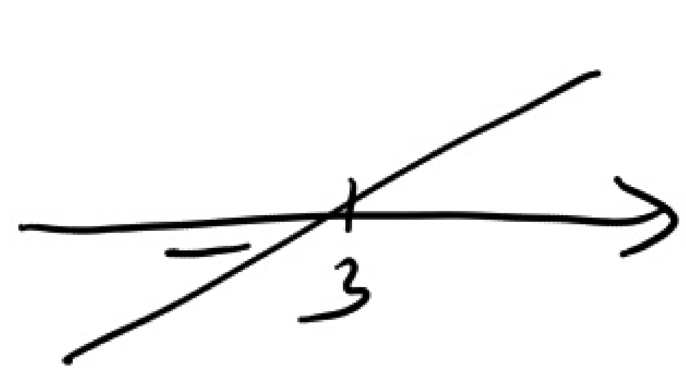
(C)

Zad. 4

$$\lim_{x\to 3^-}\frac{-x+2}{x^2-5x+6}=\lim_{x\to 3^-}\frac{-\cancel{(x-2)}}{(x-3)\cancel{(x-2)}}=$$

$$=\lim_{x\to 3^-}\frac{-1}{x-3}=\left[\frac{-1}{0^-}\right]=\infty$$

(D)



Zad. 5 ⟨2p⟩

$A = (-5; 3) \Rightarrow$ środek sym. $f(x)$

$f(x) = \dfrac{ax+7}{x+d} \wedge x \neq -d$

$\dfrac{d}{a} = ?$

① $f(x) = \dfrac{a(x+d)+7-ad}{x+d} = \dfrac{7-ad}{x+d} + a$

$y = \dfrac{7-ad}{x} \xrightarrow{[-d; a]} f(x)$

$A' = (0, 0) \xrightarrow{[-d; e]} A = (-5; 3) = (-d; a)$

$d = 5 \wedge a = 3$

② $\dfrac{d}{a} = \dfrac{5}{3} = 1,(6) = \underbrace{1,66}_{\text{odp:}}6 \ldots$

Zad. 6 ⟨3p⟩ $f(x) = \sqrt{3}x^2 - 1$

$P_0 = (x_0; y_0) \in (f \cap s)$

$s: y = ax + b$

$s: y = f'(x_0)(x - x_0) + f(x_0)$

$a = f'(x_0) = tg\alpha$

$\alpha = 30^\circ$

$P = ?$

① $f'(x) = 2\sqrt{3}x$

② $f'(x_0) = tg\, 30^\circ = \dfrac{\sqrt{3}}{3}$

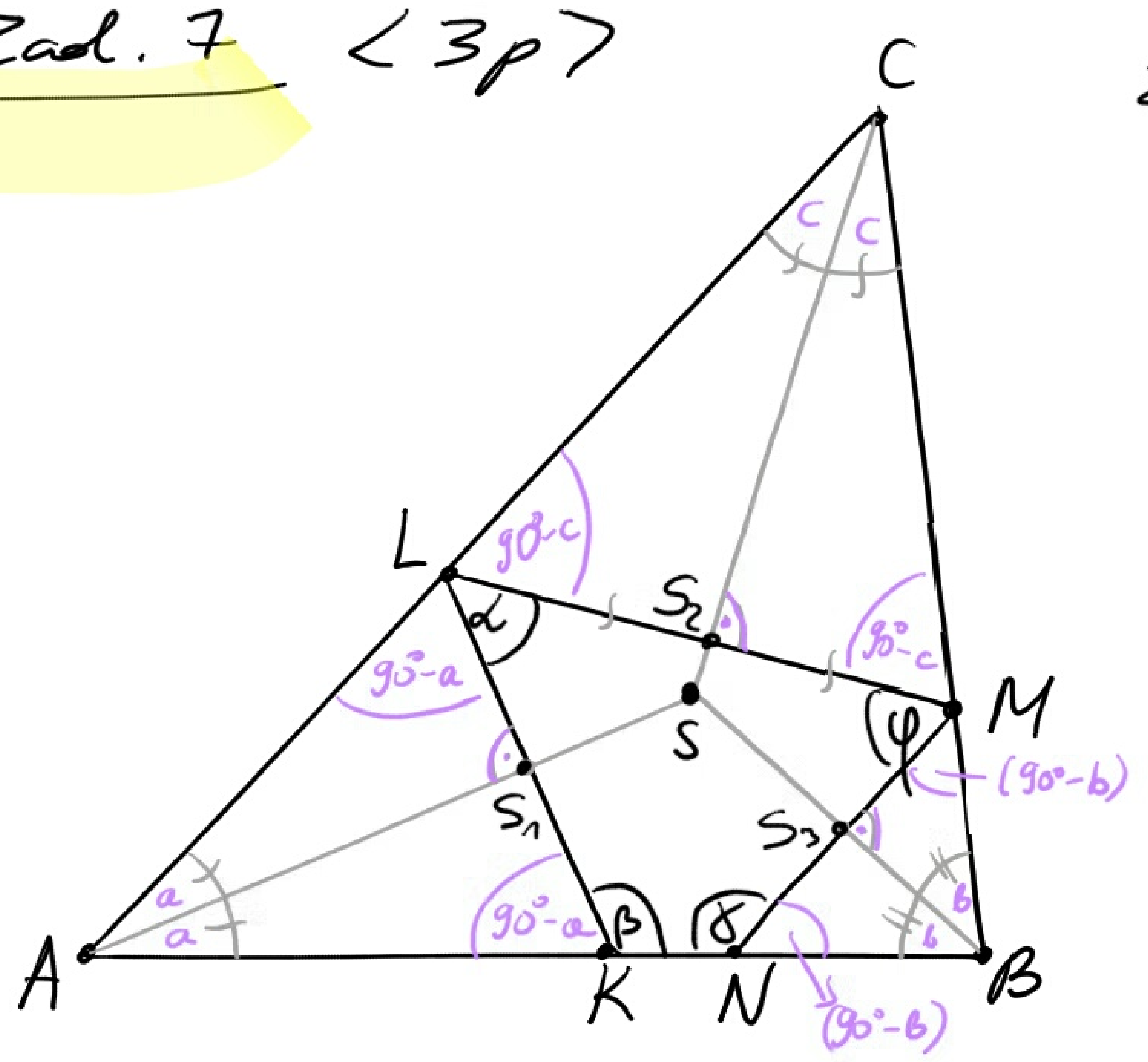
$2\sqrt{3}x_0 = \dfrac{\sqrt{3}}{3} \quad | \cdot \dfrac{1}{2\sqrt{3}}$

$\underbrace{x_0 = \dfrac{1}{6}} \longrightarrow f\left(x_0 = \dfrac{1}{6}\right) = \sqrt{3} \cdot \dfrac{1}{36} - 1$

$\underbrace{f\left(\dfrac{1}{6}\right) = \dfrac{\sqrt{3}-36}{36}} = y_0$

Odp: $P_0 = \left(\dfrac{1}{6}; \dfrac{\sqrt{3}-36}{36}\right)$

Zad. 7 <3p>



Zał:

$|AC| > |BC|$

$d_C = |KC|$

$d_B =$

$d_A =$

$|LS_2| = |S_2M|$

$|LS_1| = |S_1K|$

$|MS_3| = |S_3N|$

Teza:

$$\begin{cases} \alpha+\gamma = 180° \\ \beta+\varphi = 180° \end{cases}$$

Dowód: $(90°-a)$; $(90°-b)$; $(90°-c)$ $\Rightarrow$ KĄTY WYNIKAJĄCE Z Δ PROSTOKĄTNYCH NA RYSUNKU

① ΔABC: $2(a+b+c) = 180° \ /:2$

$$\underline{a+b+c = 90°}$$

②
$$\begin{cases} \beta = 180° - (90°-a) = 90° + a \\ \gamma = 180° - (90°-b) = 90° + b \\ \varphi = 180° - (90°-b) - (90°-c) = b+c \\ \alpha = 180° - (90-c) - (90°-a) = a+c \end{cases}$$

③ $\alpha+\gamma = a+c+90°+b = a+b+c+90° = 90°+90° = \underline{\underline{180°}}$

$\beta+\varphi = 90°+a+b+c = 90°+90° = 180°$

$$\begin{cases} \alpha+\gamma = 180° \\ \beta+\varphi = 180° \end{cases} \Rightarrow$$ NA CZWOROKĄCIE KLMN DA SIĘ OPISAĆ OKRĄG

cnd

Zad. 8 <3p>

Z: $k, m, n \in C$

$L = k^3 \cdot m - k \cdot m^3$

T: $6 | L \Rightarrow L = 6n$

D:

I METODA:

① $L = km(k^2 - m^2) = km(k^2 - 1 + 1 - m^2) =$

$= km \cdot [(k^2-1) - (m^2-1)] = km[(k-1)(k+1) - (m-1)(m+1)]$

$= km(k-1)(k+1) - km(m-1)(m+1) =$

$= m \cdot \underbrace{(k-1) \cdot k(k+1)} - k \cdot \underbrace{(m-1) \cdot m \cdot (m+1)}$

ILOCZYN TRZECH KOLEJNYCH LICZB CAŁKOWITYCH JEST PODZIELNY NA 6 - [BO W 3 KOLEJNYCH LICZBACH CAŁKOWITYCH WYSTĘPUJE MINIMUM JEDNA LICZBA PODZIELNA NA DWA ORAZ — " — " — " — " — " — TRZY]. RÓŻNICA DWÓCH LICZB PODZIELNYCH NA 6 JEST PODZIELNA NA 6. cnd.

II METODA:

① $L = km(k^2 - m^2) = km(k-m)(k+m)$

② OZNACZENIA: P - liczba parzysta, NP - liczba nieparzysta

$L = k \cdot m \cdot (k-m)(k+m)$

$k \cdot m$	$(k-m)$	$(k+m)$		
P · P	P	P	→	$2 \mid L$
NP · NP	P	P	→	$2 \mid L$
P · NP	NP	NP	→	$2 \mid L$

$\Rightarrow 2 | L$

③ dla $3 | k \lor 3 | m \to 3 | L$

dla $k = 3a+1 \land m = 3b+1$

$L = (3a+1)(3b+1)(3a+1-3b-1)(3a+1+3b+1)$

$= (3a+1)(3b+1) \cdot 3(a-b) \cdot (3a+3b+2) \to 3 | L$

dla $k = 3a+2 \land m = 3b+1$

$L = (3a+2)(3b+1)(3a+2-3b-1)(3a+2+3b+1)$

$= (3a+2)(3b+1)(3a-3b+1) \cdot 3(a+b+1) \to 3 | L$

dla $k = 3a+2 \land m = 3b+2$

$L = (3a+2)(3b+2)(3a+2-3b-2)(3a+2+3b+2)$

$= (3a+2)(3b+2) \cdot 3(a-b) \cdot (3a+3b+4) \to 3 | L$

$\Rightarrow 3 | L$

② ∧ ③ $(2 | L \land 3 | L) \Rightarrow 6 | L$

cnd

Zad. 9 <4p> $\Omega = \{1; \underline{2}; 3; \underline{4}; 5; \underline{6}; 7; 8\}$

$n = 8$

$k = 8$ (Ki, BP)

$P(A) = ?$

A – żadne dwie liczby parzyste nie sąsiadują ze sobą

① $\overline{\overline{\Omega}} = 8! = 40320$

I METODA:

② ROZWAŻAMY WSZYSTKIE PRZESTAWIENIA LICZB PARZYSTYCH (P)

$$\overline{\overline{A}} = \underset{P}{3}\ 5\ \underset{P}{2}\ 4\ \underset{P}{1}\ 3\ 2\ 1 \quad \begin{matrix} \times 4 \\ \times 3 \\ \times 2 \\ \times 1 \end{matrix}$$

$$= 2 \times 5! \cdot 3! \,(4+3+2+1) = 2 \cdot 5! \cdot 3! \cdot 10 = \underline{\underline{14400}}$$

③ $P(A) = \dfrac{\overline{\overline{A}}}{\overline{\overline{\Omega}}} = \dfrac{14400}{40320} = \underline{\underline{\dfrac{5}{14}}}$

II METODA:

② ROZWAŻAMY PRZYPADKI TRZECH LICZB PARZYSTYCH OBOK SIEBIE

$$\overline{\overline{A}}_3 = \underset{P}{3}\ \underset{P}{2}\ \underset{P}{1}\ 5\ 4\ 3\ 2\ 1 \quad \times 6 \quad = 3!\, 5! \times 6$$

③ ROZWAŻAMY PRZYPADKI DWÓCH LICZB PARZYSTYCH OBOK SIEBIE

$$\overline{\overline{A}}_2 = \underset{P}{3}\ \underset{P}{2}\ 5\ \underset{P}{1}\ 4\ 3\ 2\ 1 \times 5 \quad \times 2$$

(PP na początku oraz PP na końcu)

$$+ \quad \_\ \underset{P}{\_}\ \underset{P}{\_}\ \_\ \underset{P}{\_}\ \_\ \_\ \_ \times 4 \quad \times 5$$

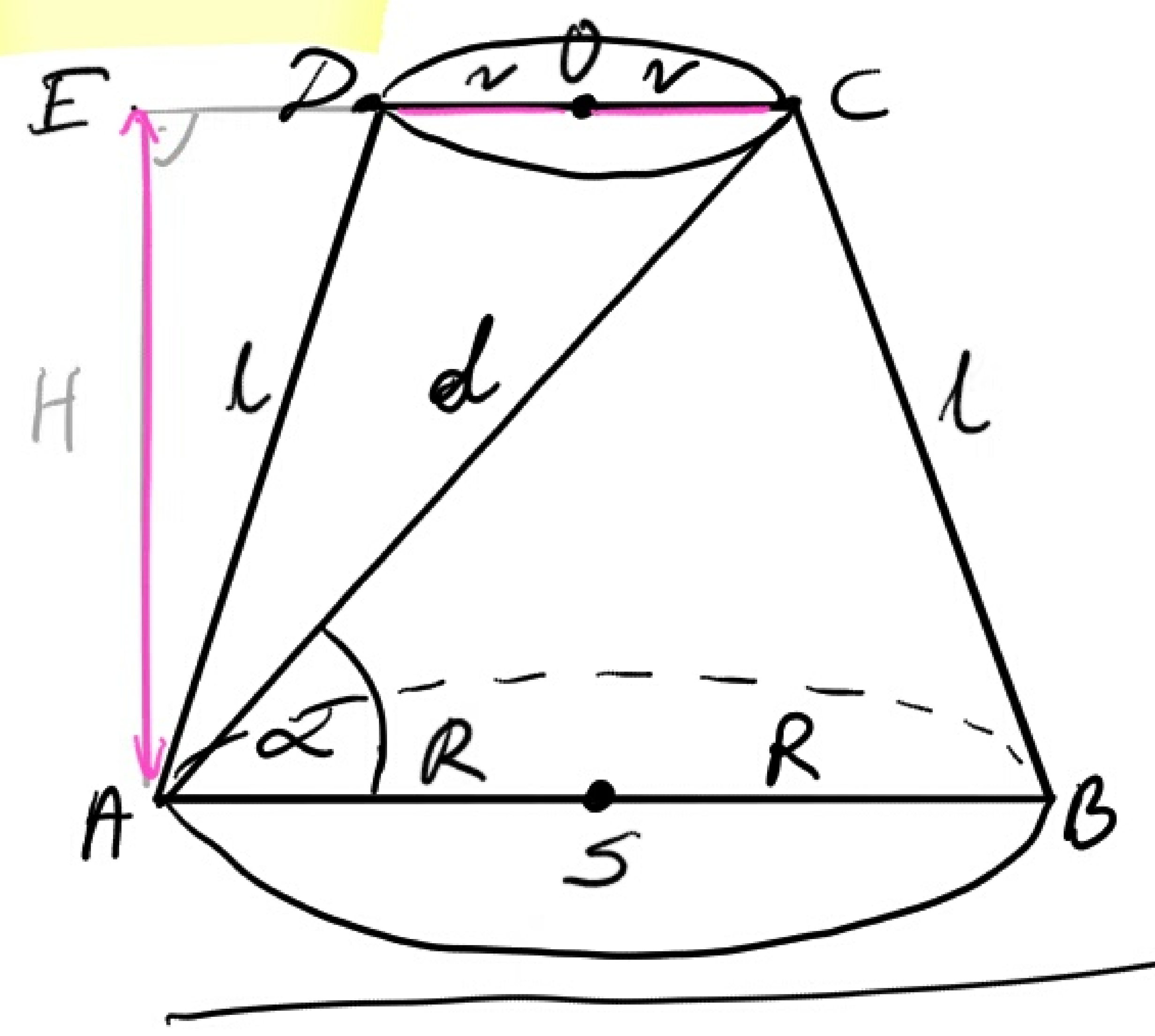
④

$$\overline{\overline{A}}' = \overline{\overline{A}}_3 + \overline{\overline{A}}_2 = 3! \cdot 5! \cdot [6 + 5 \times 2 + 4 \times 5] = 25920$$

$$\overline{\overline{A}} = \overline{\overline{\Omega}} - \overline{\overline{A}}' = 40320 - 25920 = \underline{\underline{14400}}$$

$$P(A) = 5/14$$

Zad. 10 <4p>



$V = \frac{1}{3}\pi \cdot H(r^2 + rR + R^2)$

$0 < r < R$

$H > 0$

$H = 10$

① $V = 840\pi$

$r = 6$

$d = |AC|$

$\cos\alpha = ?$

① $\frac{1}{3}\cdot\pi\cdot 10\cdot(36 + 6\cdot R + R^2) = 840\pi \quad / \cdot \frac{3}{10\pi}$

$R^2 + 6R + 36 = 252$

$R^2 + 6R - 216 = 0$

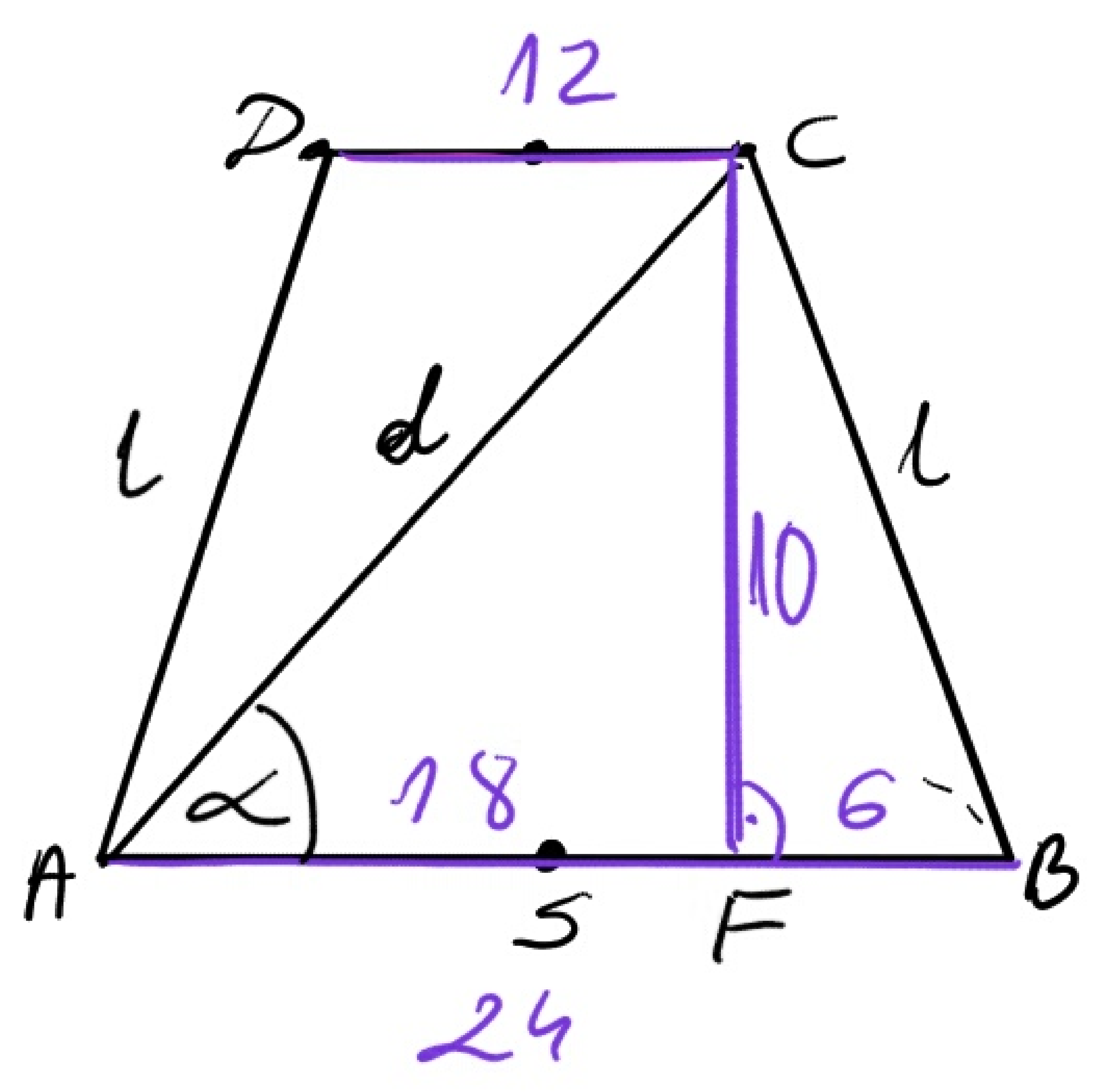
$(R+3)^2 - 9 - 216 = 0$

$(R+3)^2 = 225 \quad /\sqrt{\ } \quad \wedge \quad R > 0 \Rightarrow R + 3 > 0$

$R + 3 = 15$

$R = 12$

② $|AF| = \frac{2R + 2r}{2} = 18$



$d^2 = 18^2 + 10^2$

$d^2 = 324 + 100$

$d^2 = 424 \quad /\sqrt{\ } \quad \wedge \quad d > 0$

$\boxed{d = 2\sqrt{106}}$

③ $\cos\alpha = \frac{|AF|}{d} = \frac{\cancel{18}^{\,9}}{\cancel{2}\sqrt{106}} \cdot \frac{\sqrt{106}}{\sqrt{106}} = \frac{9\sqrt{106}}{106}$

Odp

Zad. 11 <4p>

$$\sin 6x + \cos 3x = 2\sin 3x + 1 \quad \Big| \quad x = ?$$
$$x \in \langle 0; \pi \rangle$$

$$\sin(2\cdot 3x) + \sin\left(\frac{\pi}{2} - 3x\right) = 2\sin 3x + 1$$

$$2\sin 3x \cdot \cos 3x + \cos 3x - 2\sin 3x - 1 = 0$$

$$\cos 3x(2\sin 3x + 1) - (2\sin 3x + 1) = 0$$

$$(2\sin 3x + 1)\cdot(\cos 3x - 1) = 0 \quad \wedge \quad \text{dla } k \in C$$

$2\sin 3x + 1 = 0 \quad \vee \quad \cos 3x - 1 = 0$

$2\sin 3x = -1 \; /:2 \qquad \cos 3x = 1$

$\sin 3x = -\frac{1}{2} \qquad 3x = 2k\pi \; /:3$

$3x = \pi + \frac{\pi}{6} + 2k\pi \quad \vee \quad 3x = 2\pi - \frac{\pi}{6} + 2k\pi \; /:3$

$x = \frac{7\pi}{18} + \frac{2k\pi}{3} \quad \vee \quad x = \frac{11\pi}{18} + \frac{2k\pi}{6} \quad \vee \quad x = \frac{2k\pi}{3}$

więc dla $x \in \langle 0; \pi \rangle$

Odp. $x = \left\{0; \frac{2\pi}{3}; \frac{7\pi}{18}; \frac{11\pi}{18}\right\}$

Zad. 12 <6p>

$m = 2.$

$x^2+(m+1)x-m^2+1=0$

ZR: $x_1 \neq x_2 \longrightarrow (1)\ \Delta_x > 0$

$(2)\ x_1^3+x_2^3 > -7x_1x_2$

(1) $(m+1)^2-4\cdot 1\cdot(-m^2+1)>0$

$m^2+2m+1+4m^2-4>0$

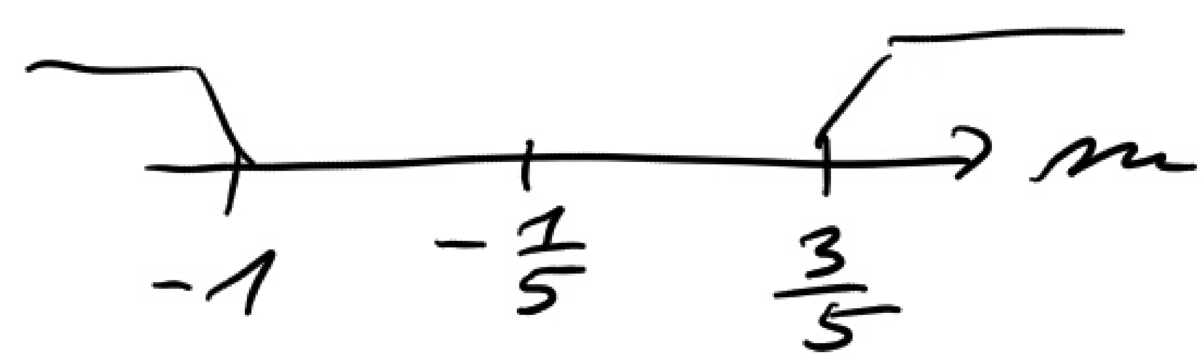
$5m^2+2m-3>0 \quad /:5 \quad \Rightarrow$ (lub z Δ)

$m^2+\frac{2}{5}m-\frac{3}{5}>0$

$(m+\frac{1}{5})^2-\frac{1}{25}-\frac{15}{25}>0$

$(m+\frac{1}{5})^2>\frac{16}{25} \quad /\sqrt{\ }$

$|m+\frac{1}{5}|>\frac{4}{5} \quad \Rightarrow$



$Z_1: \ m\in(-\infty;-1)\cup(\frac{3}{5};\infty)$

(2) $x_1^3+x_2^3>-7x_1x_2$

$(x_1+x_2)(x_1^2-x_1x_2+x_2^2)>-7x_1x_2$

$(x_1+x_2)\cdot[(x_1+x_2)^2-3x_1x_2]>-7x_1x_2$

$-(m+1)\cdot[(m+1)^2-3(-m^2+1)]>-7\overbrace{(-m^2+1)}^{-(m^2-1)}$

$-(m+1)(m^2+2m+1+3m^2-3)-7(m+1)(m-1)>0$

$-(m+1)\cdot(4m^2+2m-2)-7(m+1)(m-1)>0$

$-(m+1)[(4m^2+2m-2)+7(m-1)]>0$

$-(m+1)(4m^2+9m-9)>0$

$-4(m+1)(m^2+\frac{9}{4}m-\frac{9}{4})>0$

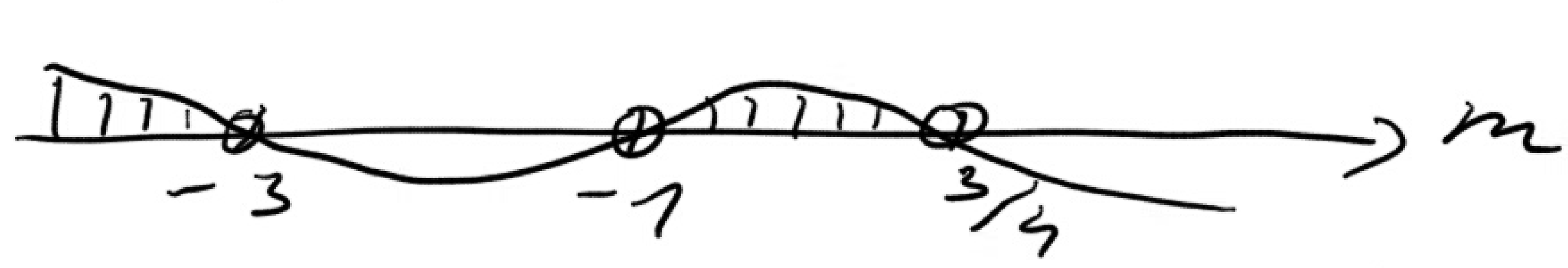
$-4(m+1)[(m+\frac{9}{8})^2-\frac{81}{64}-\frac{144}{64}]>0$

$-4(m+1)[(m+\frac{9}{8})^2-\frac{225}{64}]>0$

$-4(m+1)[(m+\frac{9}{8})^2-(\frac{15}{8})^2]>0$

$-4(m+1)(m+\frac{9}{8}-\frac{15}{8})(m+\frac{9}{8}+\frac{15}{8})>0$

$-4(m+1)(m-\frac{3}{4})(m+3)>0$



$Z_2: \ m\in(-\infty;-3)\cup(-1;\frac{3}{4})$

$Z_1 \cap Z_2$:

Odp: $m\in(-\infty;-3)\cup(\frac{3}{5};\frac{3}{4})$

Zad. 13 <4p>

$$a_n = a_1 q^{n-1}$$

① $\begin{cases} a_3 + a_6 = -84 \\ a_4 + a_7 = 168 \end{cases}$

② $S_n = 32\,769$

$n = ?$

① $\begin{cases} a_1 q^2 + a_1 q^5 = -84 \\ a_1 q^3 + a_1 q^6 = 168 \end{cases} \Rightarrow \begin{cases} a_1 q^2 (1+q^3) = -84 \\ a_1 q^3 (1+q^3) = 168 \end{cases}$

$$-84q = 168 \quad /:(-84)$$

$$\boxed{q = -2}$$

$$a_1 \cdot (-2)^2 \cdot [1 + (-2)^3] = -84$$

$$a_1 \cdot 4 \cdot (-7) = -84 \quad /:(-28)$$

$$\boxed{a_1 = 3}$$

② $a_1 \cdot \dfrac{1-q^n}{1-q} = 32769$

$$3 \cdot \frac{1-(-2)^n}{1-(-2)} = 32769$$

$$3 \cdot \frac{1-(-2)^n}{3} = 32769$$

$$1 - (-2)^n = 32769$$

$$-(-2)^n = 32768 \quad /(-1)$$

$$(-2)^n = -32768$$

$$(-2)^n = (-2)^{15}$$

Odp. $\boxed{n = 15}$

Zad. 14 <6 p>

$B=(x_B; y_B)=?$

$C=(x_C; y_C)=?$

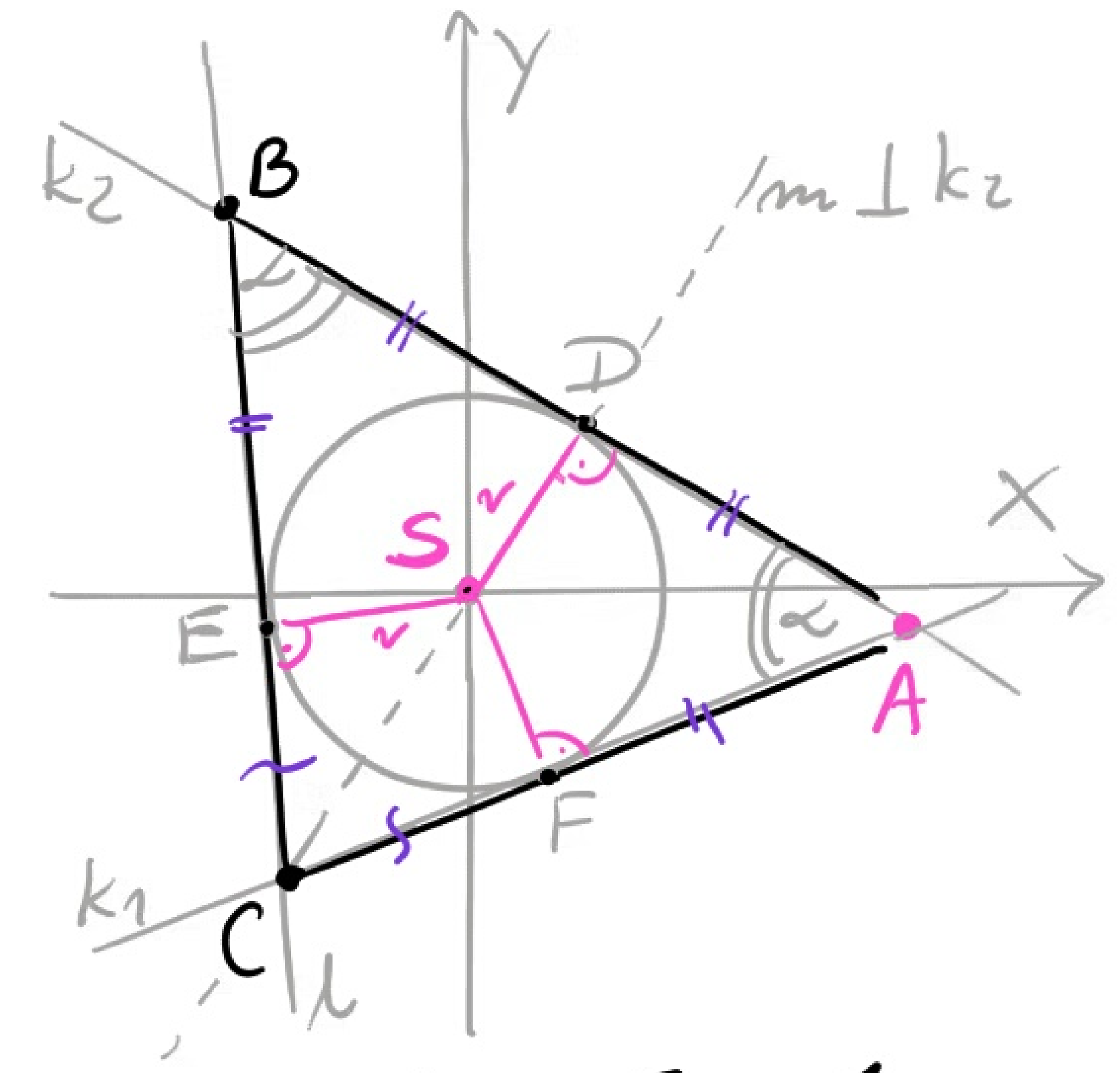
$|AC|=|BC|$

$A=(7;-1)$

$\{x_C; y_C\}\in \mathbb{R}_-$

$O_1:\ x^2+y^2=10$

$\hookrightarrow$ okrąg wpis. w ΔABC



① $O_1:\ x_1^2+x_2^2=10$

$\boxed{S=(0;0)\ ;\ r=\sqrt{10}}$

② $A\in k: y=ax+b \Rightarrow$

$-1=7a+b \Rightarrow \boxed{b=-7a-1}$

$k:\ y=ax-7a-1$

$\boxed{k:\ ax-y-7a-1=0}$

oraz: $k_1\nearrow \ \wedge \ C;F\in k_1$

$k_2\searrow \ \wedge \ B;D\in k_2$

③ $r=d_{(S,k)}:\ \sqrt{10}=\frac{|0-0-7a-1|}{\sqrt{a^2+1}}\ /\cdot\sqrt{a^2+1}$

$\sqrt{10(a^2+1)}=|7a+1|\ /^2$

$10a^2+10=49a^2+14a+1$

$39a^2+14a-9=0$

$\Delta=1600\ ;\ \sqrt{\Delta}=40$

$a_2=\frac{-14-40}{2\cdot 39}=-\frac{9}{13}$

$a_1=\frac{-14+40}{2\cdot 39}=\frac{1}{3}$

$\Rightarrow$

$k:\ y=ax-7a-1$

$k_2:\ y=-\frac{9}{13}x+\frac{50}{13}$

$k_1:\ y=\frac{1}{3}x-\frac{10}{3}$

④ $D\in k_2\cap O_1$:

$\begin{cases} y=-\frac{9}{13}x+\frac{50}{13} \\ x^2+y^2=10 \end{cases}$

$x^2+\left(\frac{50}{13}-\frac{9}{13}x\right)^2=10$

$x^2+\frac{1}{169}(50-9x)^2=10\ /\cdot 169$

$169x^2+2500-900x+81x^2=1690$

$250x^2-900x+810=0\ /:10$

$25x^2-90x+81=0$

$(5x-9)^2=0$

$\begin{cases} x=\frac{9}{5} \\ y=\frac{13}{5} \end{cases} \Rightarrow \boxed{D=\left(\frac{9}{5};\frac{13}{5}\right)}$

⑤ $\overrightarrow{AD}=\overrightarrow{DB}$

$\left[\frac{9}{5}-7;\frac{13}{5}+1\right]=\left[x-\frac{9}{5};y-\frac{13}{5}\right]$

$\begin{cases} x_B=\frac{9}{5}-7+\frac{9}{5}=-\frac{17}{5}=-3\frac{2}{5} \\ y_B=\frac{13}{5}+1+\frac{13}{5}=\frac{31}{5}=6\frac{1}{5} \end{cases}$

$\boxed{B=\left(-3\frac{2}{5};6\frac{1}{5}\right)}$

⑥ $m:$ $\begin{array}{l} S: \\ D: \end{array}\begin{cases} b=0 \\ \frac{9}{5}a+b=\frac{13}{5} \end{cases}$

$a=\frac{13}{5}\cdot\frac{5}{9}=\frac{13}{9}$

$m:\ y=\frac{13}{9}x$

⑦ $C\in(m\cap k_1)$:

$\begin{array}{l} m: \\ k_1: \end{array}\begin{cases} y=\frac{13}{9}x \\ y=\frac{1}{3}x-\frac{10}{3} \end{cases} \Rightarrow$

$\frac{13}{9}x=\frac{1}{3}x-\frac{10}{3}\ /\cdot 9$

$13x=3x-30$

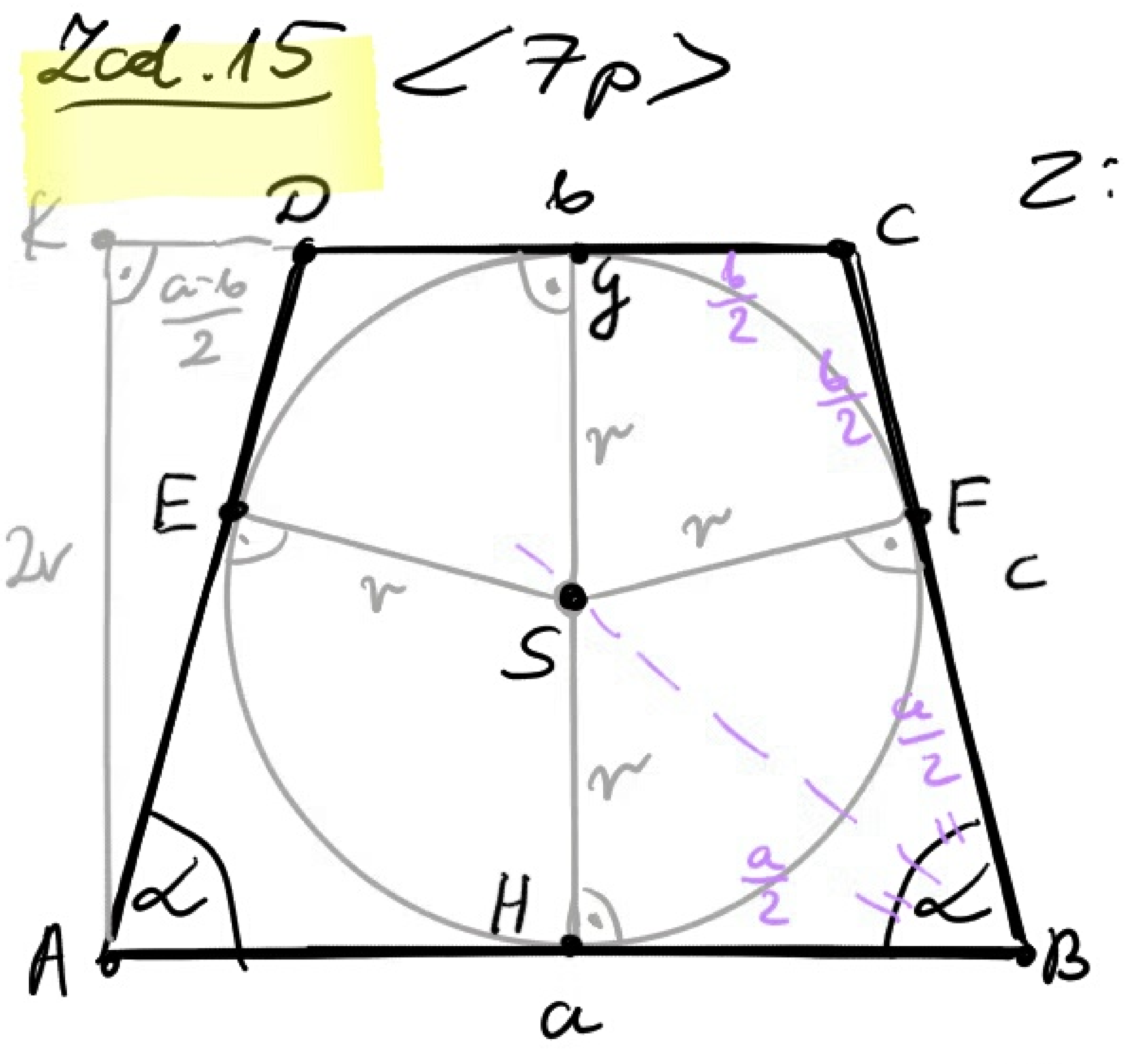
$10x=-30$

$x=-3$

$y=\frac{13}{9}\cdot(-3)=-\frac{13}{3}=-4\frac{1}{3}$

$\boxed{C=\left(-3;-4\frac{1}{3}\right)}$

Zad. 15 <7p>



D:

Z: $a=|AB|>0$

$c=|AD|=|BC|>0$

$b=|DC|$

$a+2r=2$

b) $L=a+b+2c$

c) $L=min$

Sz:

a) $a=?$

b) T: $L(a)\overset{?}{=}\frac{4a^2-8a+8}{a}$

c) $tg\alpha=?$

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① $a+b=2c \Rightarrow \underline{c=\frac{a+b}{2}}>0 \quad \wedge \quad a+2r=2 \Rightarrow r=\frac{2-a}{2}>0$

$2-a>0$

$0<\underline{a<2}$

$\Downarrow$

$\underline{a\in(0;2)}$

$c^2=(2r)^2+\left(\frac{a-b}{2}\right)^2 \Rightarrow c^2=4r^2+\frac{1}{4}(a-b)^2$

$\frac{(a+b)^2}{4}=4\cdot\frac{(2-a)^2}{4}+\frac{(a-b)^2}{4} \quad /\cdot 4$

$\cancel{a^2}+2ab+\cancel{b^2}=4(4-4a+a^2)+\cancel{a^2}-2ab+\cancel{b^2}$

$4ab=16-16a+4a^2 \quad /:(4a)>0$

$\underline{b=\frac{a^2-4a+4}{a}}=a-4+\frac{4}{a}=\frac{(a-2)^2}{a}>0$

oraz

$\frac{a-b}{2}>0 \Rightarrow a>b \Rightarrow a>\frac{(a-2)^2}{a} \Rightarrow a^2>a^2-4a+4$

$4a>4 \quad /:4$

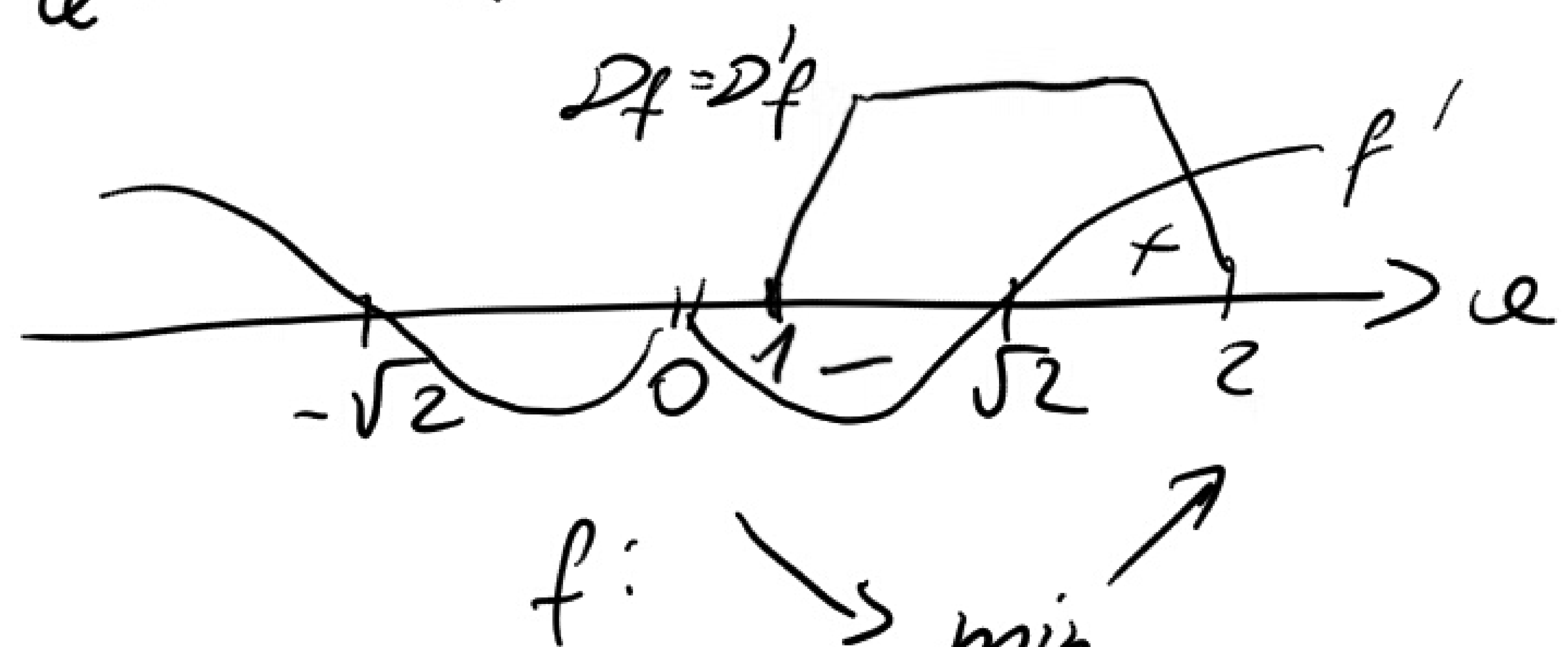
$\underline{a>1}$

Ad. a) $\underline{a\in(1;2)}$

② $L=2(a+b)=2\cdot\left(a+\frac{a^2-4a+4}{a}\right)=2\cdot\frac{2a^2-4a+4}{2}$

Ad. b) $\underline{L(a)=\frac{4a^2-8a+8}{a}}$ cnd.

③ $L(a)=4a-8+\frac{8}{a} \Rightarrow L'(a)=4-\frac{8}{a^2}=\frac{4a^2-8}{a^2}=\frac{4(a-\sqrt{2})(a+\sqrt{2})}{a^2}$



$L'(a)=0$ dla $a=\sqrt{2}$

$L'(a)<0$ dla $a\in(1;\sqrt{2})$

$L'(a)>0$ dla $a\in(\sqrt{2};2)$

$\Downarrow$

dla $\underline{a=\sqrt{2}}$ funkcja posiada minimum lokalne będące również najmniejszą wartością.

④ $2r=\underline{2-\sqrt{2}}$; $\frac{a-b}{2}=\frac{\sqrt{2}-(\sqrt{2}-4+\frac{4}{\sqrt{2}})}{2}=\frac{4-2\sqrt{2}}{2}=\underline{2-\sqrt{2}}$

Ad. c) $tg\alpha=\frac{2r}{\frac{a-b}{2}}=\frac{2-\sqrt{2}}{2-\sqrt{2}}=1$