Zad. 1 <1pkt>

$$(2\sqrt{3}x+4y)^3 = (2\sqrt{3}x)^3 + 3\cdot(2\sqrt{3}x)^2\cdot 4y + \underbrace{3\cdot 2\sqrt{3}x(4y)^2}_{\underbrace{6\sqrt{3}\cdot 4^2}_{96\sqrt{3}}\, xy^2} + (4y)^3$$

C

Zad. 2 <1pkt>

$W(x) = 6x^3+3x^2-5x+p$

$W(1)=0$

$p = ?$

$6+3-5+p=0 \Rightarrow \underline{p=-4}$

D

Zad. 3 <1pkt>

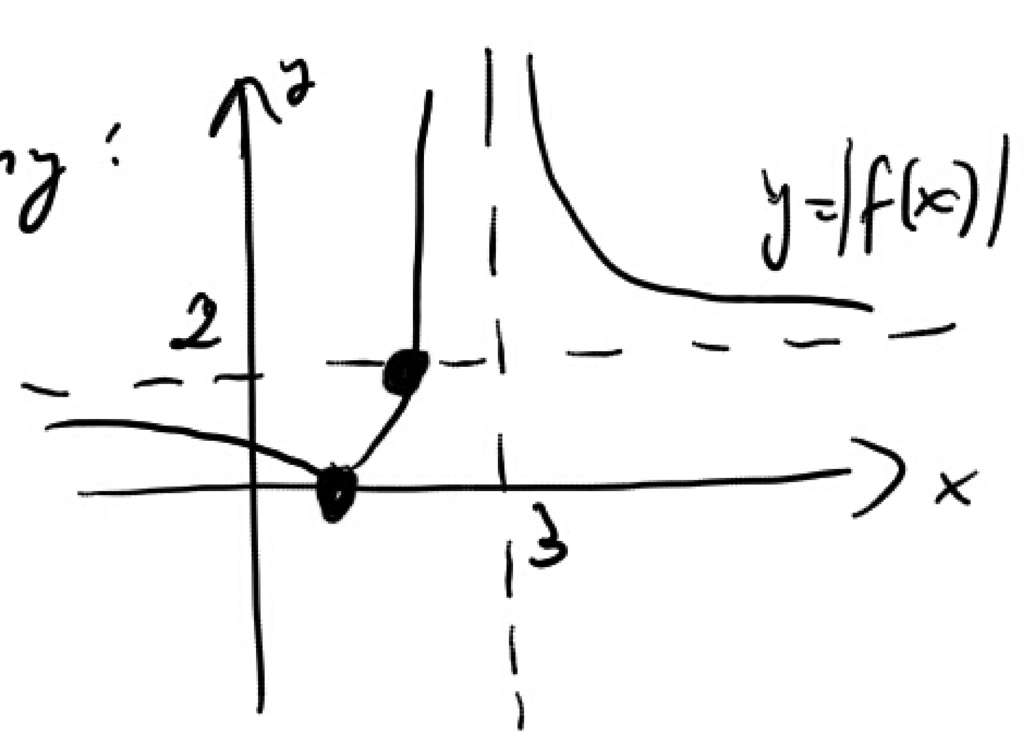
$f(x) = \frac{a}{x-b} + c \Rightarrow$ dany jest wykres $f(x)$

$D_f = (-\infty;3)\cup(3;\infty)$

$|f(x)| = p$ ma 1 rozwiąz.

$p = ?$

z wykresu mamy: $y=|f(x)|$



$|f(x)|=p$ ma 1 rozwiązanie dla

$\underline{p=\{0;2\}}$

B

Zad. 4 <1pkt>

$f(x) = \frac{3x-1}{x^2+4}$

$f'(x) = ?$

A

$$f'(x) = \frac{3(x^2+4)-(3x-1)\cdot 2x}{(x^2+4)^2} = \frac{3x^2+12-6x^2+2x}{(x^2+4)^2} = \underline{\frac{-3x^2+2x+12}{(x^2+4)^2}}$$

Zad. 5 <1pkt>

$$\lim_{n\to\infty} \frac{(pn^2+4n)^3}{5n^6-4} = -\frac{8}{5}$$

$p = ?$

$$\lim_{n\to\infty} \frac{(pn^2+4n)^3}{5n^6-4} = \frac{p^3}{5} = -\frac{8}{5} \Rightarrow \underline{p=-2}$$

D

Zad. 6 <2 pkt.>

$\overline{\overline{\Omega}} = 10000 = n$

$k = 1$

A - wybrana osoba popiera budowy

B - wybrana osoba jest mężczyzną

$P(A|B) = ?$

$\overline{\overline{A}} = 5140 + 2260 = 7400$

$\overline{\overline{B}} = 2260 + 740 = 3000$

$\overline{\overline{A \cap B}} = 2260$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\overline{\overline{A\cap B}}}{\overline{\overline{B}}}$$

$$P(A|B) = \frac{2260}{3000} = \frac{113}{150} = 0{,}75(3)$$

7	5	3

Zad. 7 <2 pkt>

$a_n = a_1 q^{n-1} = \left(\frac{1}{2x-371}\right)^n$

$n \in N^+ \wedge x \in C$

$a_n > 0$

$a_1 + a_2 + \ldots = S \Rightarrow |q| < 1$

$x_{min} = ?$

$$q = \frac{a_{n+1}}{a_n} = \left(\frac{1}{2x-371}\right)^{n+1-n} = \frac{1}{2x-371}$$

$$\begin{cases} |q| < 1 \\ a_n > 0 \Rightarrow 0 < q < 1 \end{cases}$$

dla $q > 0$

$0 < \frac{1}{2x-371} < 1 \quad / \cdot (2x-371) > 0$

$2x > 371$

$x > 185{,}5$

$0 < 1 < 2x - 371$

$2x > 372$

$x > 186 \wedge x \in C$

$\boxed{x_{min} = 187}$

Zad. 8 ⟨3 pkt⟩

Z: $x, y \in \mathbb{R} \wedge x, y > 0$

$x^2+y^2=2$

T: $x+y \le 2 \quad /()^2$ dla $x+y>0$

D: I metoda

1) $(x-y)^2 \ge 0 \Rightarrow x^2+y^2-2xy \ge 0 \Rightarrow \underbrace{x^2+y^2 \ge 2xy}$

(2) $(x+y)^2 = x^2+y^2+\underbrace{2xy} \le x^2+y^2+\underbrace{x^2+y^2} = 2(x^2+y^2)$

$(x+y)^2 \le 2(x^2+y^2) \wedge x^2+y^2=2$

$(x+y)^2 \le 2 \cdot 2$

$(x+y)^2 \le 4 \quad /\sqrt{\ }$ dla $x, y > 0$

$\boxed{x+y \le 2}$ cnd.

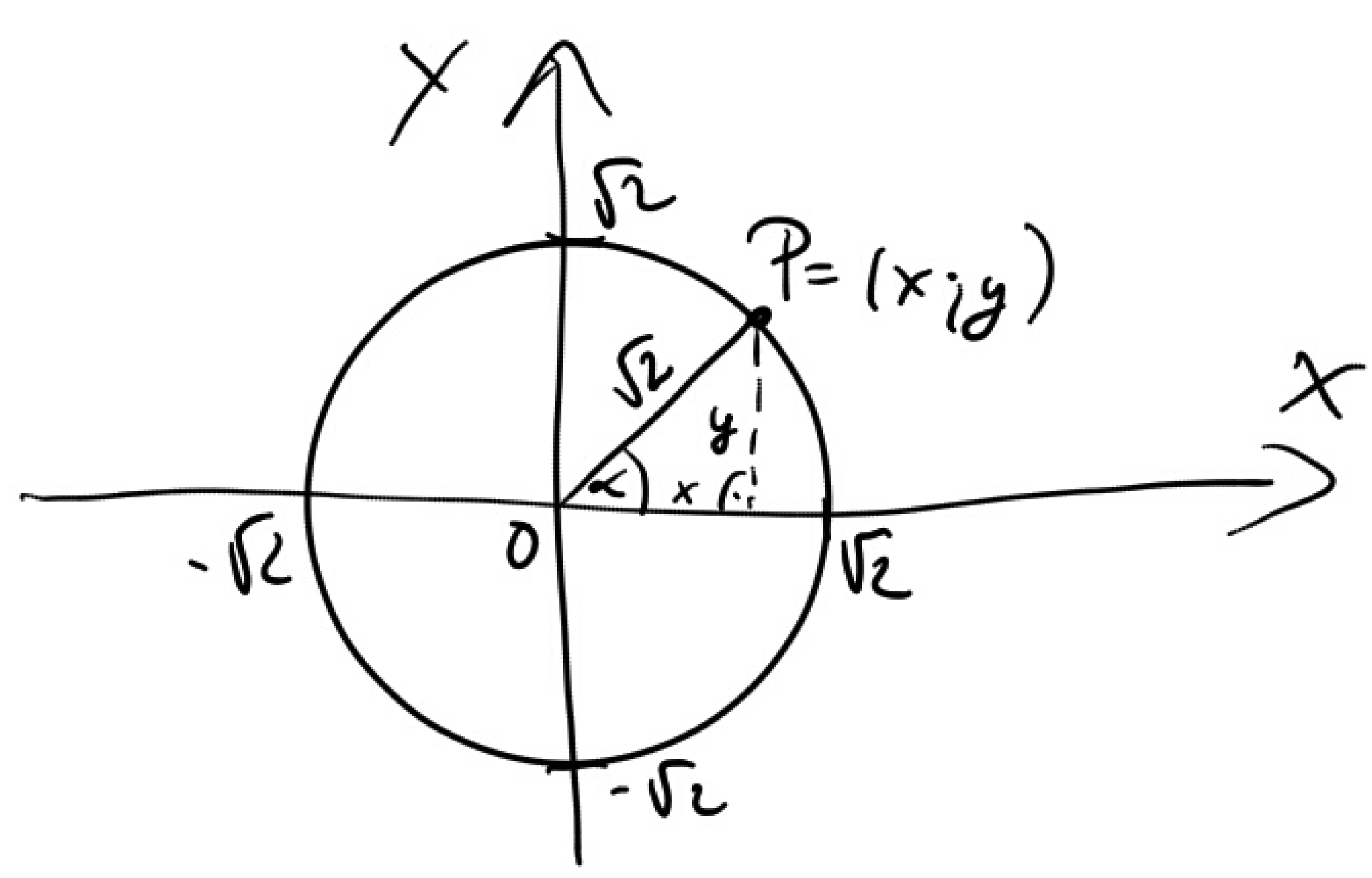
II metoda

O_1: $x^2+y^2=2 \rightarrow$ okrąg O_1: $r=\sqrt{2} \wedge S=(0,0)$

$P \in O_1$ dla $\alpha \in (0^\circ; 360)$

$\sin\alpha = \frac{y}{\sqrt{2}} \Rightarrow y = \sqrt{2}\cdot\sin\alpha$

$\cos\alpha = \frac{x}{\sqrt{2}} \Rightarrow x = \sqrt{2}\cos\alpha$



więc:

$x+y = \sqrt{2}\sin\alpha + \sqrt{2}\cos\alpha$

$x+y = \sqrt{2}(\sin\alpha + \cos\alpha)$

$x+y = \sqrt{2}\,[\sin\alpha + \sin(90-\alpha)]$

$x+y = \sqrt{2}\cdot 2\cdot \sin\frac{\alpha+90-\alpha}{2}\cdot\cos\frac{\alpha-90+\alpha}{2}\,]$

$x+y = 2\sqrt{2}\cdot\sin 45^\circ\cdot\cos(\alpha-45^\circ)$

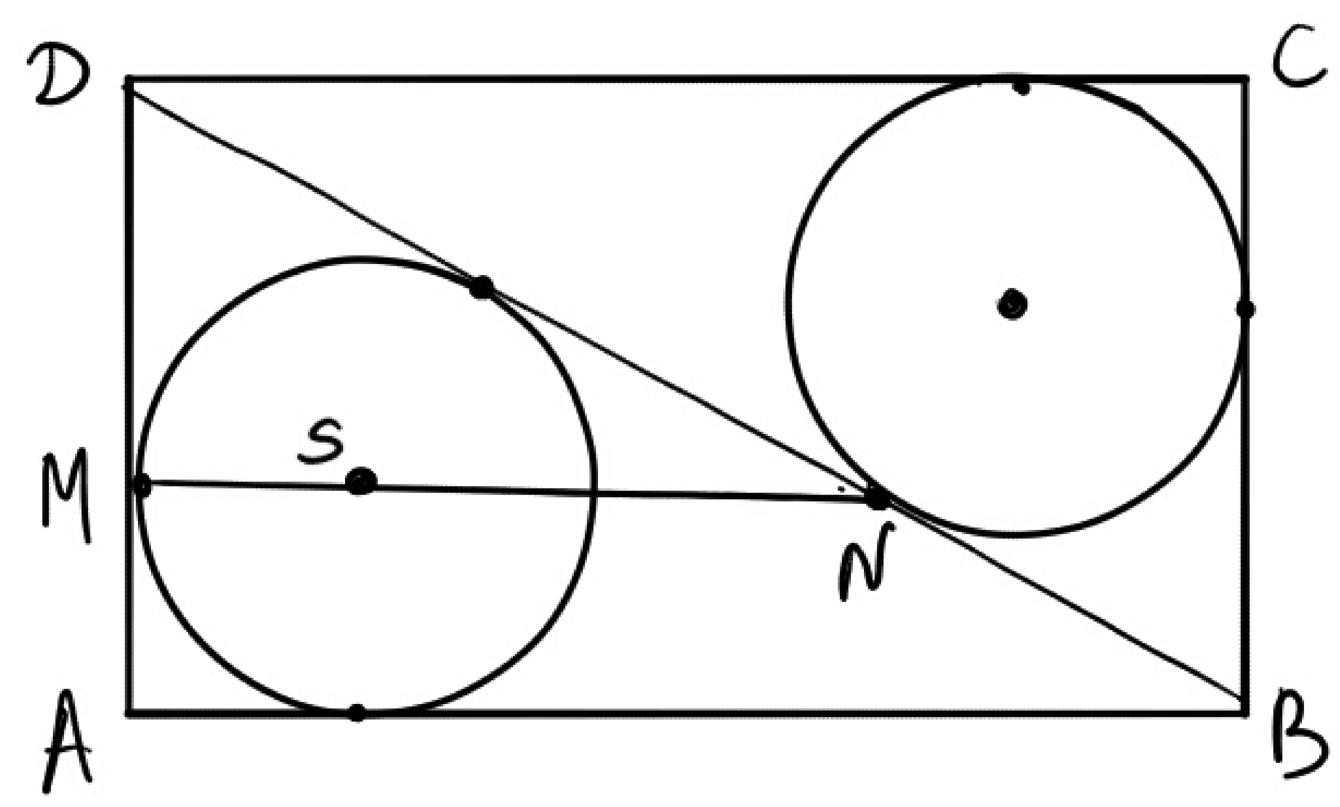
$x+y = 2\sqrt{2}\cdot\frac{\sqrt{2}}{2}\cdot\cos(\alpha-45^\circ)$

$x+y = 2\cdot\cos(\alpha-45^\circ) \le 2$ (bo $\cos\beta \in \langle -1;1\rangle$)

$x+y \le 2$ cnd.

Zad. 9 (3 pkt)

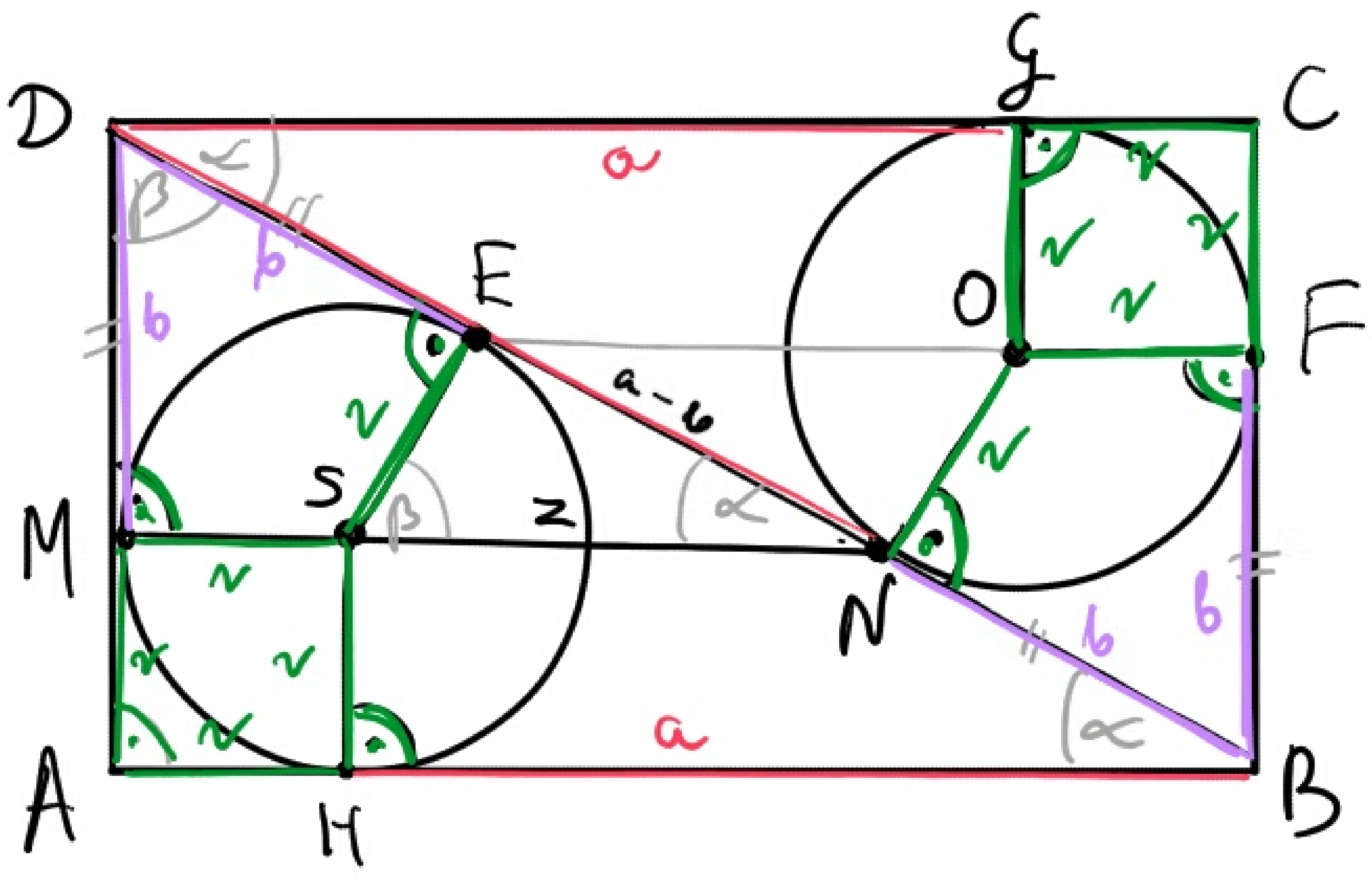
Z:



T:

$$|MN| \stackrel{?}{=} |AD|$$

D:



(1) <u>OZNACZENIA i WŁASNOŚCI:</u>

r – promień okręgu

$r = |SM| = |SE| = |SH| = |AM| = |AH| = |ON| = |OG| = |OF| = |GC| = |FC|$

$y = |EN| \;\wedge\; z = |SN| = |EO|$

$r + z = |MN| = |EF|$

z własności DELTOIDU:

$b = |DM| = |DE| = |NB| = |FB|$

$a = |HB| = |DG| = |DN| = |EB|$

(2) $\alpha + \beta = 90^\circ$

$\Delta ABD \sim \Delta MND \sim \Delta SNE$ z własności kkk $(90^\circ, \alpha, \beta)$

$$\operatorname{tg}\alpha = \frac{b+r}{a+r} = \frac{b}{z+r} = \frac{r}{a-b}$$

$$r(a+r) = (a-b)(b+r)$$

$$\cancel{ar} + r^2 = ab + \cancel{ar} - b^2 - br$$

$$r(b+r) = b(a-b)$$

$$a-b = \frac{r(b+r)}{b} \Rightarrow \frac{r}{a-b} = \frac{r\cdot b}{r(b+r)} = \frac{b}{b+r}$$

(3)

$$\frac{b}{z+r} = \frac{r}{a-b} \Rightarrow \frac{b}{z+r} = \frac{b}{b+r}$$

$$\Downarrow$$

$$z+r = b+r$$

$$|MN| = |AD|$$ c.n.d.

Zad. 10 <4pkt>

$f(x) = x - 2$

$g(x) = 5 - ax$

$A = (x_A ; y_A) \in (f \cap g)$

$x_A ; y_A > 0$

$a = ?$

$x - 2 = 5 - ax$

$x + ax = 7$

$x(a+1) = 7 \quad /:(a+1) > 0 \quad$ dla $x > 0$

$x = \frac{7}{a+1} > 0 \longrightarrow$

$a + 1 > 0$

$\boxed{a > -1}$

$f(x) > 0$

$x - 2 > 0$

$\frac{7}{a+1} - \frac{2(a+1)}{a+1} > 0$

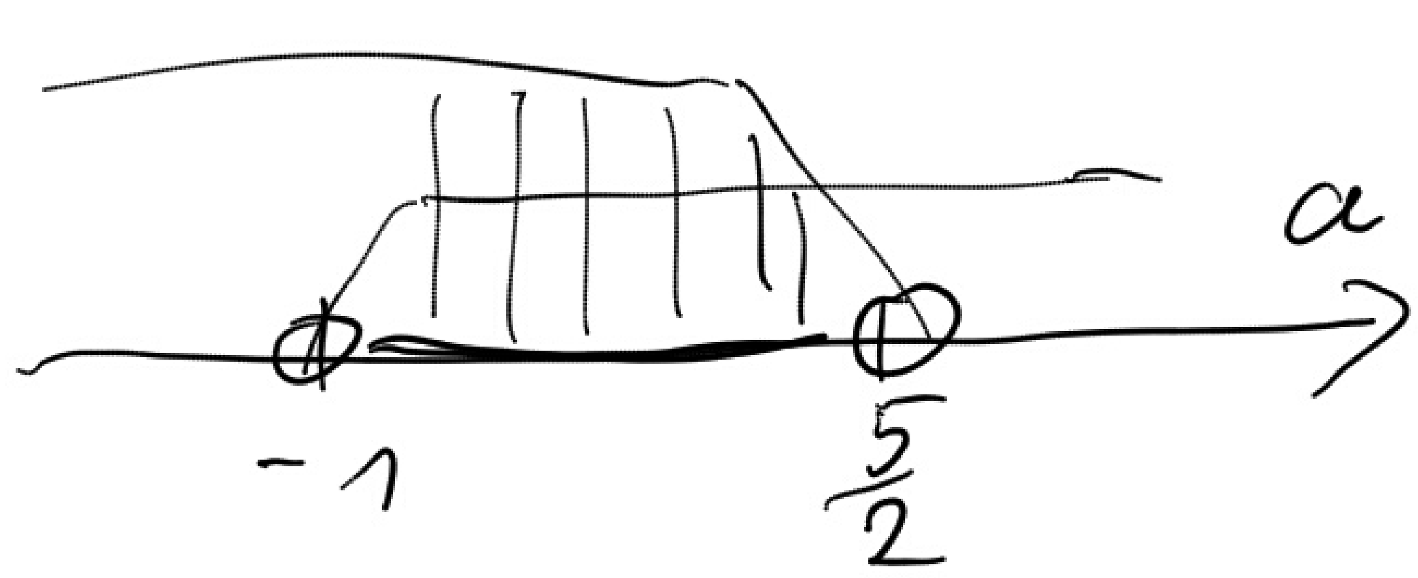
$\frac{7 - 2a - 2}{a+1} > 0$

$\frac{-2a + 5}{a+1} > 0 \quad /\cdot (a+1) > 0$

$-2a + 5 > 0$

$-2a > -5 \quad /:(-2)$

$\boxed{a < \frac{5}{2}}$



Odp:

$$\boxed{a \in \left(-1; 2\tfrac{1}{2}\right)}$$

Zad. 11 ⟨4pkt⟩

$$\frac{2\cos x - \sqrt{3}}{\cos^2 x} < 0 \quad \text{dla} \quad x \in \langle 0; 2\pi \rangle$$

1) zał: $\cos^2 x \neq 0 \Rightarrow \cos x \neq 0 \Rightarrow x \neq \frac{\pi}{2} + k\pi$

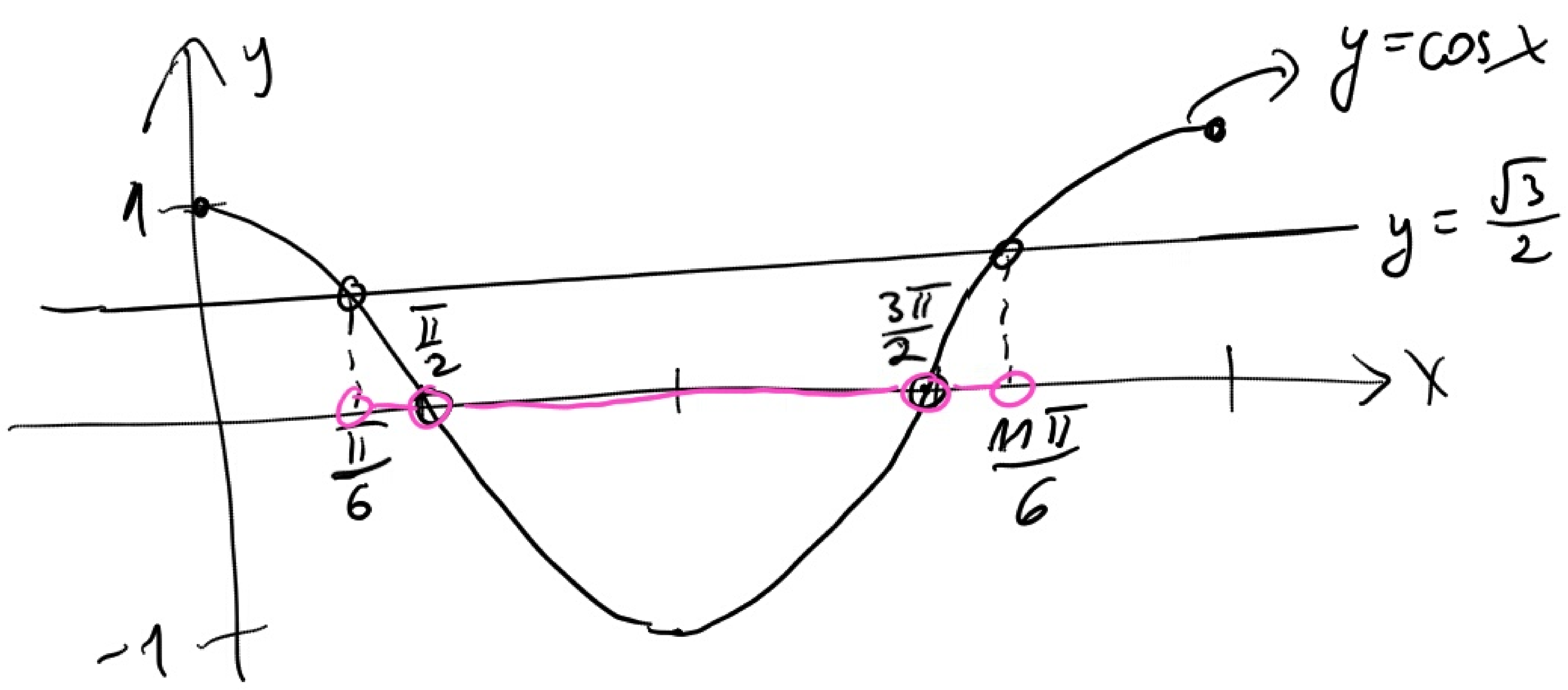
2) $$\frac{2\cos x - \sqrt{3}}{\cos^2 x} < 0 \quad / \cdot \cos^2 x > 0$$

$2\cos x - \sqrt{3} < 0$

$2\cos x < \sqrt{3} \quad /:2$

$$\cos x < \frac{\sqrt{3}}{2} \Rightarrow \left\{ \begin{array}{lll} \cos x = \frac{\sqrt{3}}{2} & \wedge & k \in C \\ x_1 = \pi + 2k\pi & \vee & x_2 = 2\pi - \frac{\pi}{6} + 2k\pi \\ x_1 = \frac{\pi}{6} + 2k\pi & \vee & x_2 = \frac{11\pi}{6} + 2k\pi \end{array} \right\}$$

dla $x \in \langle 0; 2\pi \rangle$



Odp: $x \in \left(\frac{\pi}{6}; \frac{11\pi}{6} \right) \setminus \left\{ \frac{\pi}{2}; \frac{3\pi}{2} \right\}$.

Zad. 12 (6 pkt)

$f(x) = x^2 + 2(m+1)x + 6m + 1$ $\quad$ $m = ?$

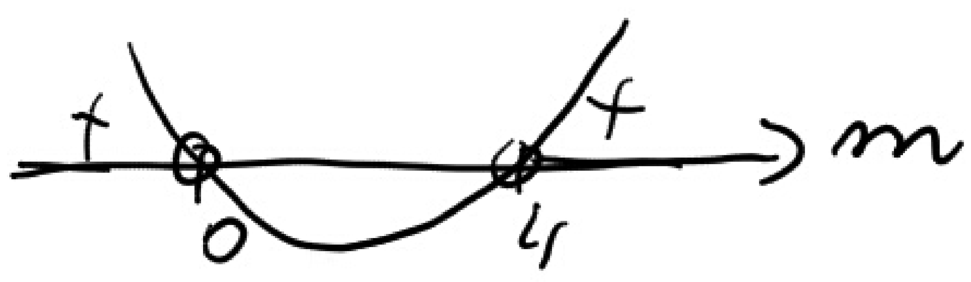
$$\begin{cases} ① \ \Delta_x > 0 \\ ② \ x_1 \cdot x_2 > 0 \\ ③ \ |x_1 - x_2| < 3 \end{cases}$$

① $4(m+1)^2 - 4(6m+1) > 0$

$4[m^2 + 2m + 1 - 6m - 1] > 0$

$4(m^2 - 4m) > 0$

$4m(m-4) > 0$



$\underline{z_1: \ m \in (-\infty; 0) \cup (4; \infty)}$

② $6m + 1 > 0$

$6m > -1 \quad /:6$

$m > -\frac{1}{6}$

$\underline{z_2: \ m \in (-\frac{1}{6}; \infty)}$

③ $|x_1 - x_2| < 3 \quad /()^2$

$x_1^2 - 2x_1x_2 + x_2^2 < 9$

$(x_1 + x_2)^2 - 4x_1x_2 < 9$

$[-2(m+1)]^2 - 4 \cdot (6m+1) < 9$

$4(m^2 + 2m + 1) - 24m - 4 - 9 < 0$

$4m^2 + 8m + 4 - 24m - 13 < 0$

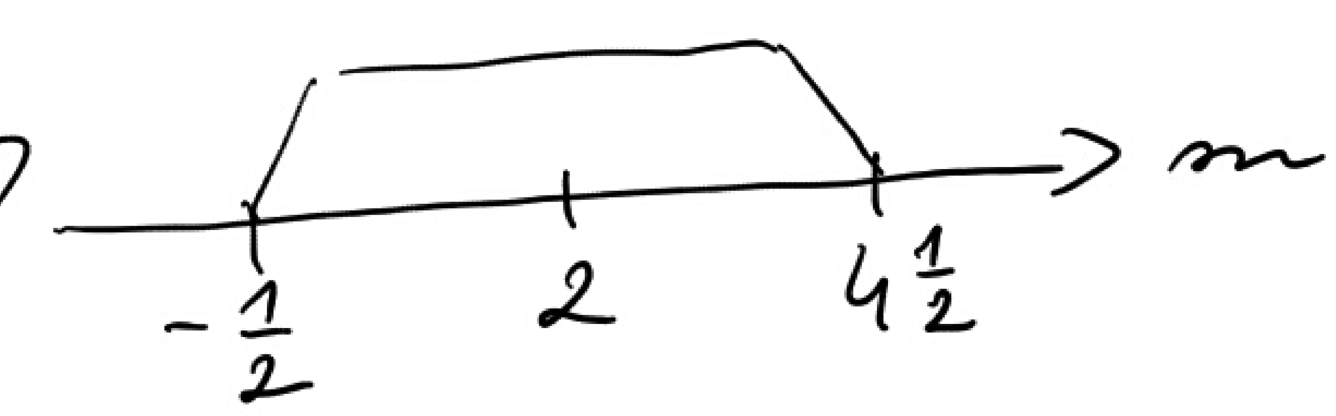
$4m^2 - 16m - 9 < 0 \quad /:4$

$m^2 - 4m - 9/4 < 0$

$(m-2)^2 - 4 - \frac{9}{4} < 0$

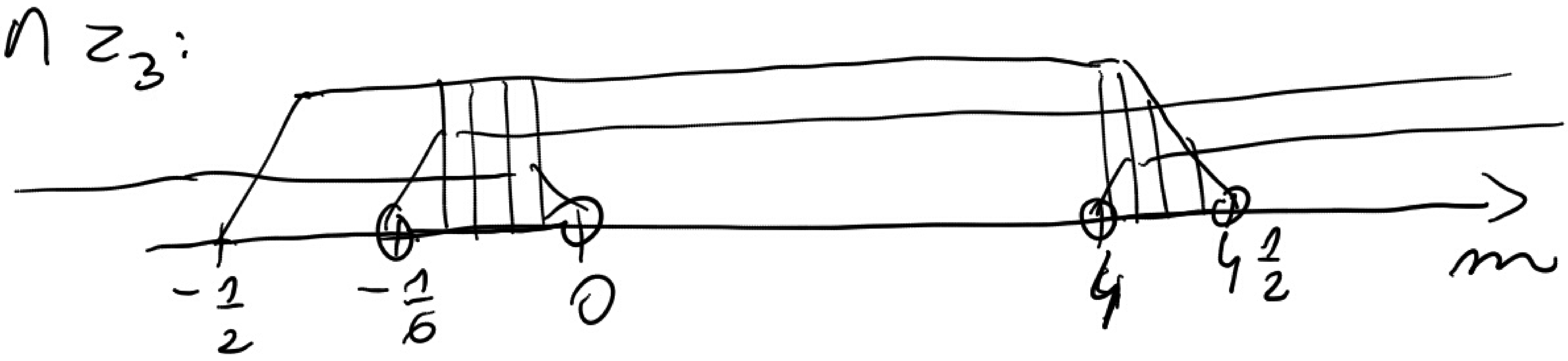
$(m-2)^2 < \frac{25}{4} \quad /\sqrt{\ }$

$|m - 2| < \frac{5}{2}$



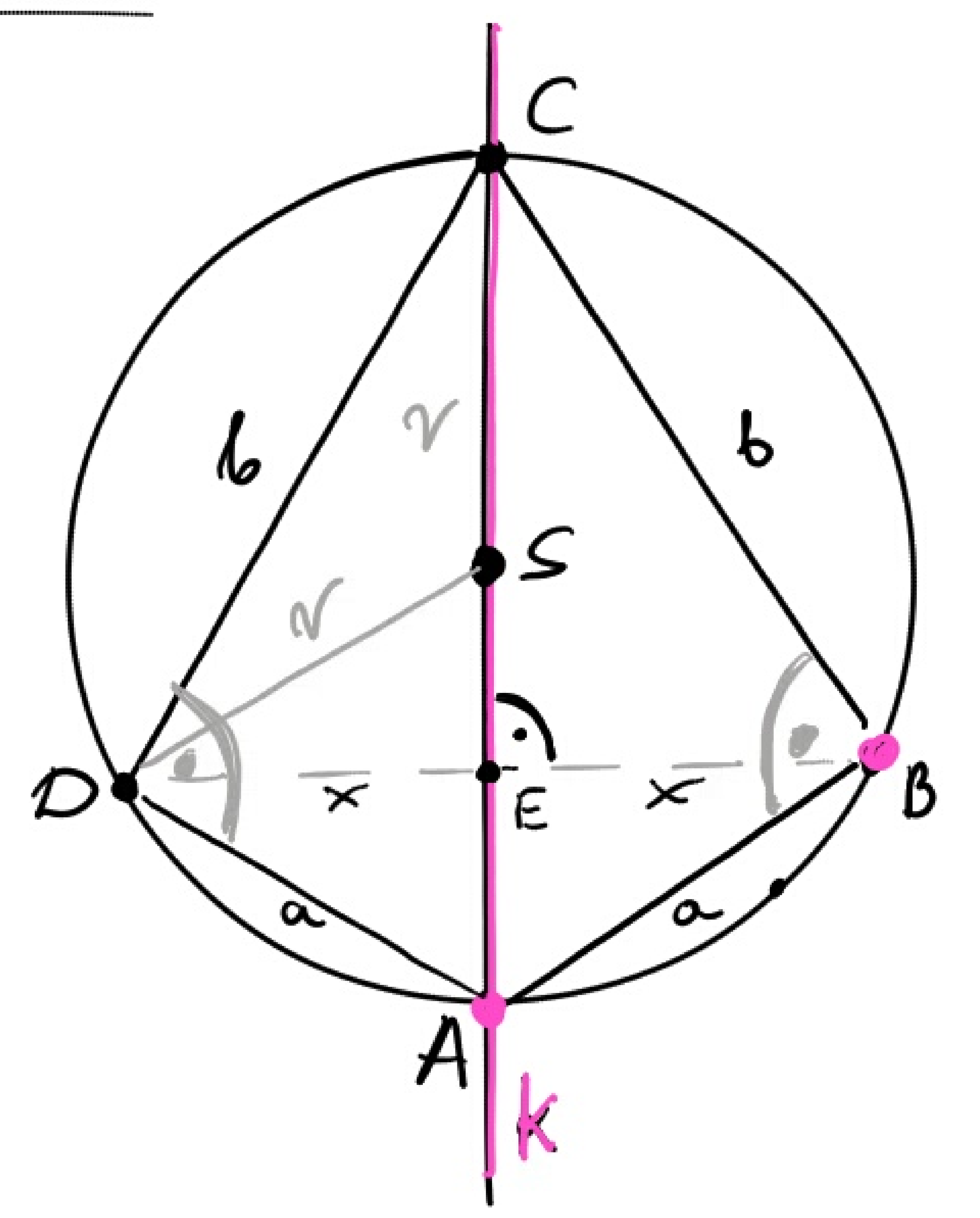
$\underline{z_3: \ m \in (-\frac{1}{2}; 4\frac{1}{2})}$

④ $z_1 \cap z_2 \cap z_3$:



Odp: $\underline{m \in (-\frac{1}{6}; 0) \cup (4; 4\frac{1}{2})}$

Zad. 13 (5 pkt)



D:

$A=(30;32)$

$B=(0;8)$

$k: x-y+2=0$

Sz:

$C=(x_C;y_C)=?$

$D=(x_D;y_D)=?$

OZNACZENIA:

$r=|AS|=|SC|=|SD|=|SB|$

$|AC|=2r$; $a=|AD|=|AB|$

$x=|DE|=|EB|$; $b=|CD|=|CB|$

① l_{AB}: $\begin{cases} 30a_1+b_1=32 \\ 0a_1+b_1=8 \end{cases}$ $\begin{cases} 30a_1=24 \quad /:30 \\ b_1=8 \end{cases}$ $\begin{cases} a_1=4/5 \\ b_1=8 \end{cases}$ $\underline{l_{AB}: y=\frac{4}{5}x+8}$

② $l_{BC} \perp l_{AB}$; $y=-\frac{5}{4}x+b$

$B \in l_{BC}$ $\quad 8=b \Rightarrow \underline{l_{BC}: y=-\frac{5}{4}x+8}$

③ $\begin{cases} l_{BC}: y=-\frac{5}{4}x+8 \\ k: x-y+2=0 \end{cases} \Rightarrow x+\frac{5}{4}x-8+2=0$

$\frac{9}{4}x=6 \quad /\cdot\frac{4}{9}$

$\Rightarrow \begin{cases} x=\frac{8}{3}=x_C \\ y_C=-\frac{5}{\cancel{4}}\cdot\frac{\cancel{8}^{2}}{3}+8=\frac{14}{3} \end{cases}$

$\boxed{C=\left(\frac{8}{3};\frac{14}{3}\right)=\left(2\frac{2}{3};4\frac{2}{3}\right)}$

④

$l_{BD} \perp k$: $\quad k: y=x+2 \Rightarrow l_{BD}: y=-x+b$

$B \in l_{BD}$: $\quad 8=0+b$

$b=8$

$\underline{l_{BD}: y=-x+8}$

⑤

$\begin{cases} k: x-y+2=0 \\ l_{BD}: y=-x+8 \end{cases} \Rightarrow x+x-8+2=0$

$2x=6 \quad /:2$

$\begin{cases} x=3 \\ y=5 \end{cases}$

$\underline{E=(3;5)}$

⑥

$\vec{DE}=\vec{EB} \Rightarrow [3-x_D; 5-y_D]=[0-3;8-5]=[-3;3]$

$\begin{cases} 3-x_D=-3 \\ 5-y_D=3 \end{cases}$ $\begin{cases} x_D=6 \\ y_D=2 \end{cases} \Rightarrow \underline{D=(6;2)}$

Odp: $C=\left(2\frac{2}{3};4\frac{2}{3}\right)$; $D=(6;2)$

Zad. 14 <3 pkt.>

Założenie:

$\Omega = \{1; 2; 3\}$

$n = 3$

$k = 10$

KOLEJNOŚĆ ISTOTNA

POWTÓRZENIA

$\Bigg\} n^k$

A – cyfra „1" występuje dokładnie trzy razy

TEZA:

$\overline{\overline{A}} = 15360$

$$\overline{\overline{A}} = \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{2}{\substack{2\\3}} \cdot \frac{2}{\substack{2\\3}} \cdot \frac{2}{\substack{2\\3}} \cdot \frac{2}{\substack{2\\3}} \cdot \frac{2}{\substack{2\\3}} \cdot \frac{2}{\substack{2\\3}} \cdot \frac{2}{\substack{2\\3}} \times (120) = \underline{\underline{2^7 \cdot 120}}$$

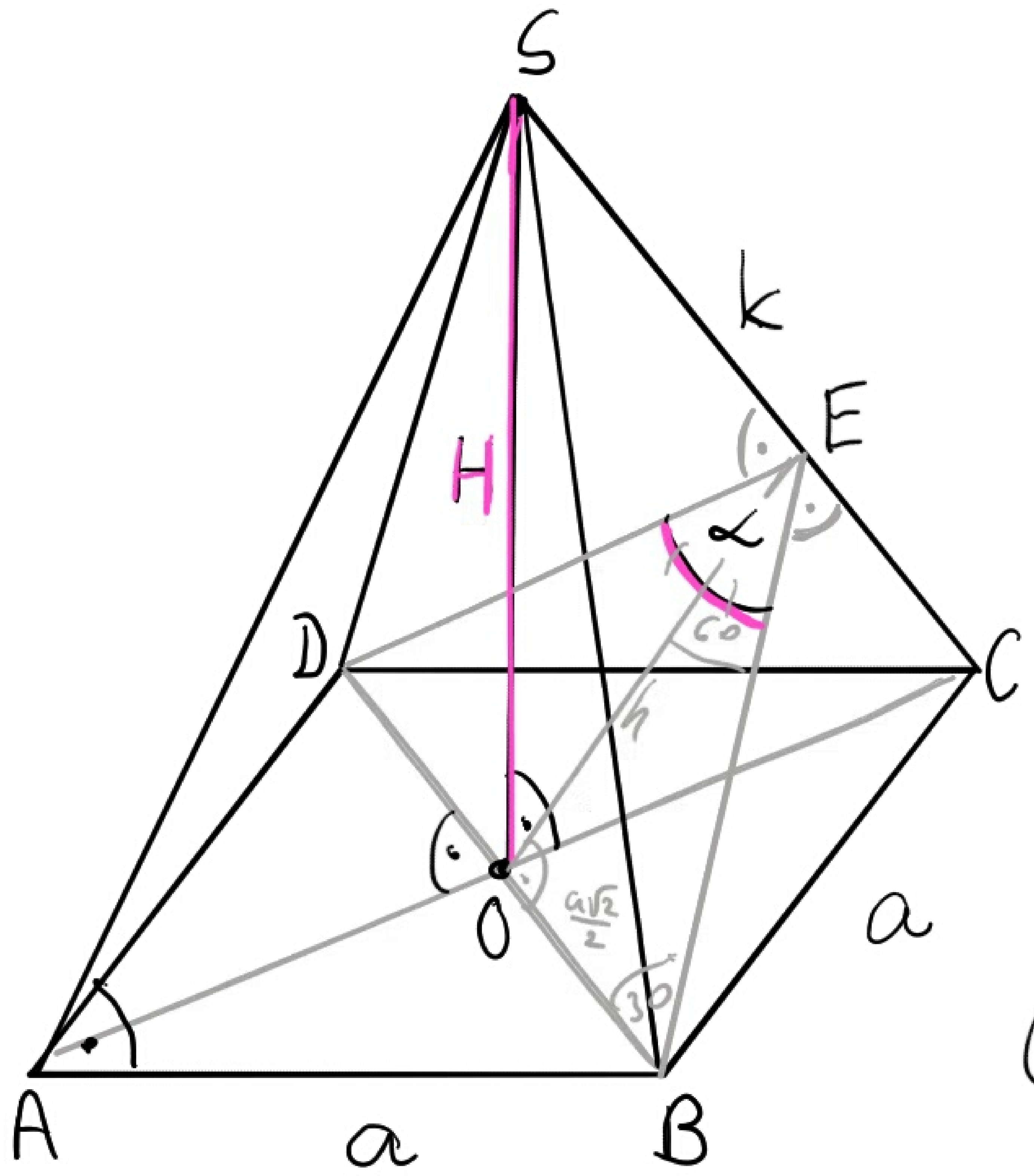
$\Uparrow$

Możliwe przestawienia: $\binom{10}{3} \cdot \binom{7}{7} = \frac{10!}{3! \cdot 7!} = \frac{\cancel{8}^{4} \cdot \cancel{9}^{3} \cdot 10}{\cancel{2} \cdot \cancel{3}} = \underline{120}$

$$\overline{\overline{A}} = 2^7 \cdot 120 = 128 \cdot 120 = 15360$$

$\underline{\underline{\text{cnd}}}$.

Zad. 15 <6 pkt>



D:

$H = |SO| = 5$

$\alpha = 120^\circ$

Sz:

$V = ?$

OZNACZENIA:

$|DE| = |BE| = x > 0$

$|OE| = h > 0$

$|AB| = |BC| = |CD| = |DA| = a > 0$

$|AS| = |BS| = |CS| = |DS| = k > 0$

(1) $\triangle DBE \; (30^\circ, 60^\circ, 90^\circ)$

$h\sqrt{3} = \frac{a\sqrt{2}}{2} \Rightarrow \boxed{h = \frac{a\sqrt{6}}{6}}$

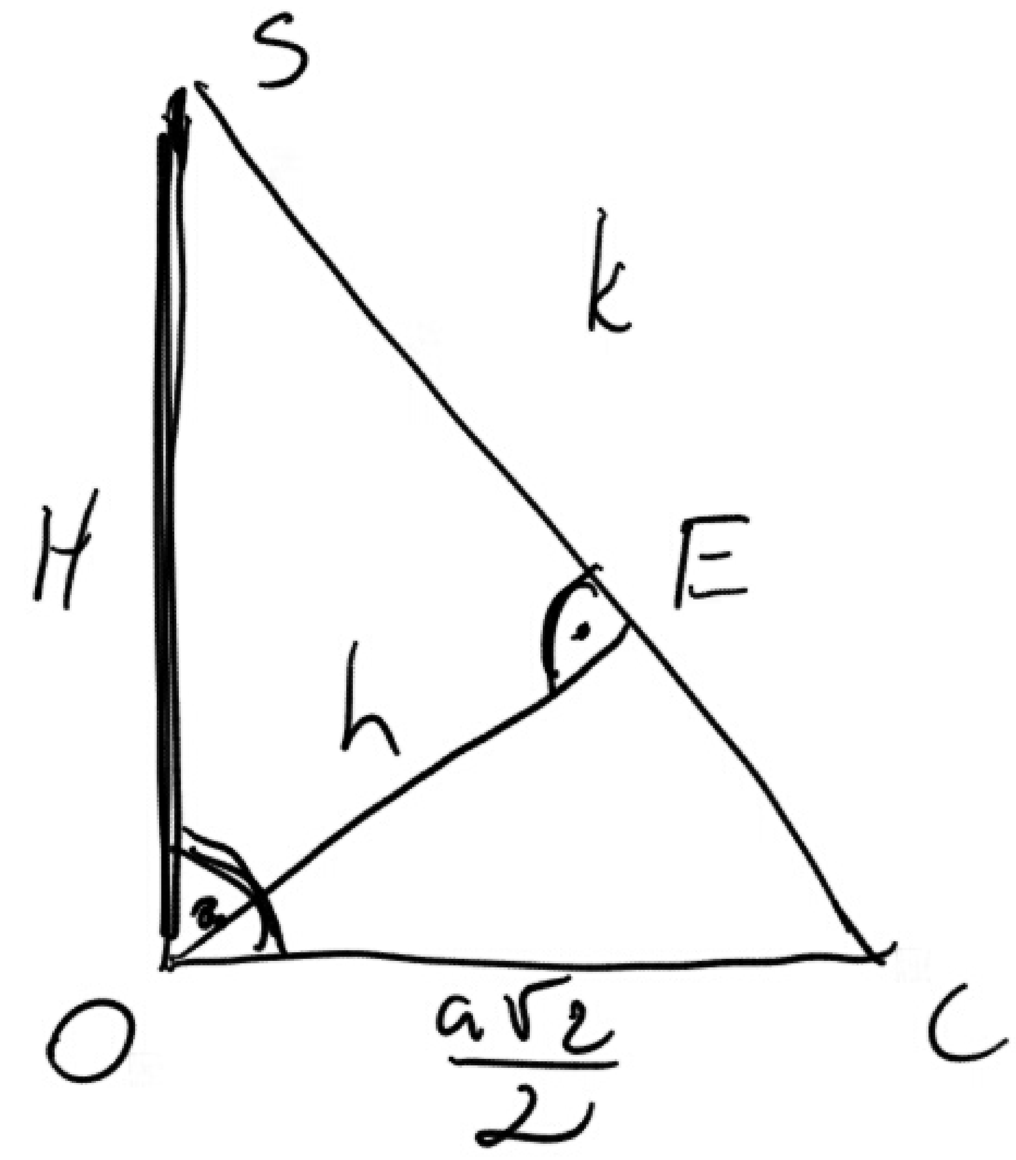
$\boxed{x = 2h = \frac{a\sqrt{6}}{3}}$

(2) $\triangle OCS$:

$k^2 = H^2 + |OC|^2$

$k^2 = 25 + \left(\frac{a\sqrt{2}}{2}\right)^2$

$k^2 = 25 + \frac{a^2}{2} \Rightarrow \boxed{k = \sqrt{\frac{a^2+50}{2}}}$



(3) $\frac{1}{2}|OC| \cdot H = \frac{1}{2}|CS| \cdot h \quad / \cdot 2$

$\frac{a\sqrt{2}}{2} \cdot 5 = \sqrt{\frac{a^2+50}{2}} \cdot \frac{a\sqrt{6}}{6} \quad / \cdot \frac{2}{a\sqrt{2}}$

$5 = \sqrt{\frac{a^2+50}{2}} \cdot \frac{\sqrt{3}}{3} \quad / \cdot \sqrt{3}$

$5\sqrt{3} = \sqrt{\frac{a^2+50}{2}} \quad / ()^2$

$75 = \frac{a^2+50}{2} \quad / \cdot 2$

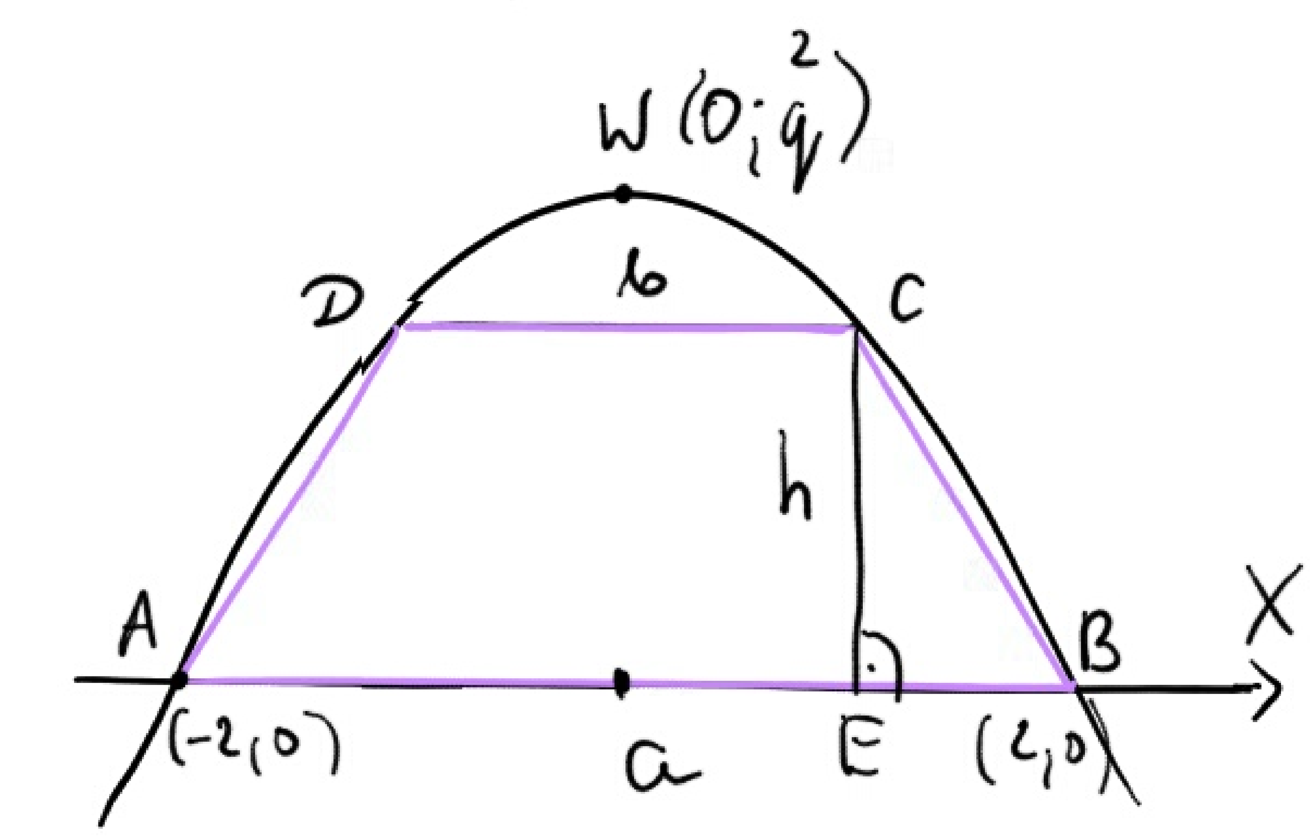
$150 = a^2 + 50$

$a^2 = 100 \quad / \sqrt{} \wedge a > 0 \Rightarrow \boxed{a = 10}$

(4) $V = \frac{1}{3} P_p \cdot H = \frac{1}{3} \cdot a^2 \cdot H = \frac{1}{3} \cdot 100 \cdot 5 = \frac{500}{3} = 166\frac{2}{3}$

Odp: Objętość ostrosłupa wynosi 166,(6) [j³].

Zad. 16 <7pkt>



$f(x) = 2 - \frac{1}{2}x^2$

$A = (-2;0) \in f(x)$

$B = (2;0) \in f(x)$

$|AD| = |BC| = c$

$\overline{AB} \parallel \overline{DC}$ i $h = |CE|$

$a = |AB| > |DC| = b$

$P_{\square} = \max$

$P(x_c) = P_{ABCD} = P_{\square} = ?$

$C = (x_c; y_c) = ?$

zał: $a, b, c, h > 0$

1) $g(x) = -\frac{1}{2}x^2 + 2 = -\frac{1}{2}(x^2-4) = -\frac{1}{2}(x-2)(x+2) \Rightarrow \begin{cases} p = x_w = 0 \\ q = y_w = 2 \end{cases}$

2) $D, C \in f(x)$, $\overline{DC} \parallel \overline{AB}$ $\Rightarrow$ $C = (x_c; 2 - \frac{1}{2}x_c^2) \wedge x_c > 0$

$D = (x_D; 2 - \frac{1}{2}x_D^2) = (-x_c; 2 - \frac{1}{2}x_c^2)$

3) $a = |-2| + |2| = 4 \wedge \overrightarrow{CD} = [-2x_c; 0] \Rightarrow b = |CD| = \sqrt{4x_c^2}$

$b = 2|x_c| = 2x_c$ dla $x_c > 0$

$l_{AB}: y = 0 \Rightarrow h = d_{C, l_{AB}} = \frac{|0 \cdot x_c + 1 \cdot (2 - \frac{1}{2}x_c^2) + 0|}{\sqrt{0^2+1^2}} = \frac{|2 - \frac{1}{2}x_c^2|}{1} \in (0;2)$

dla $x_c, h \in (0;2) \Rightarrow h = 2 - \frac{1}{2}x^2$

4) dla $x_c = x \wedge x \in (0;2)$, mamy:

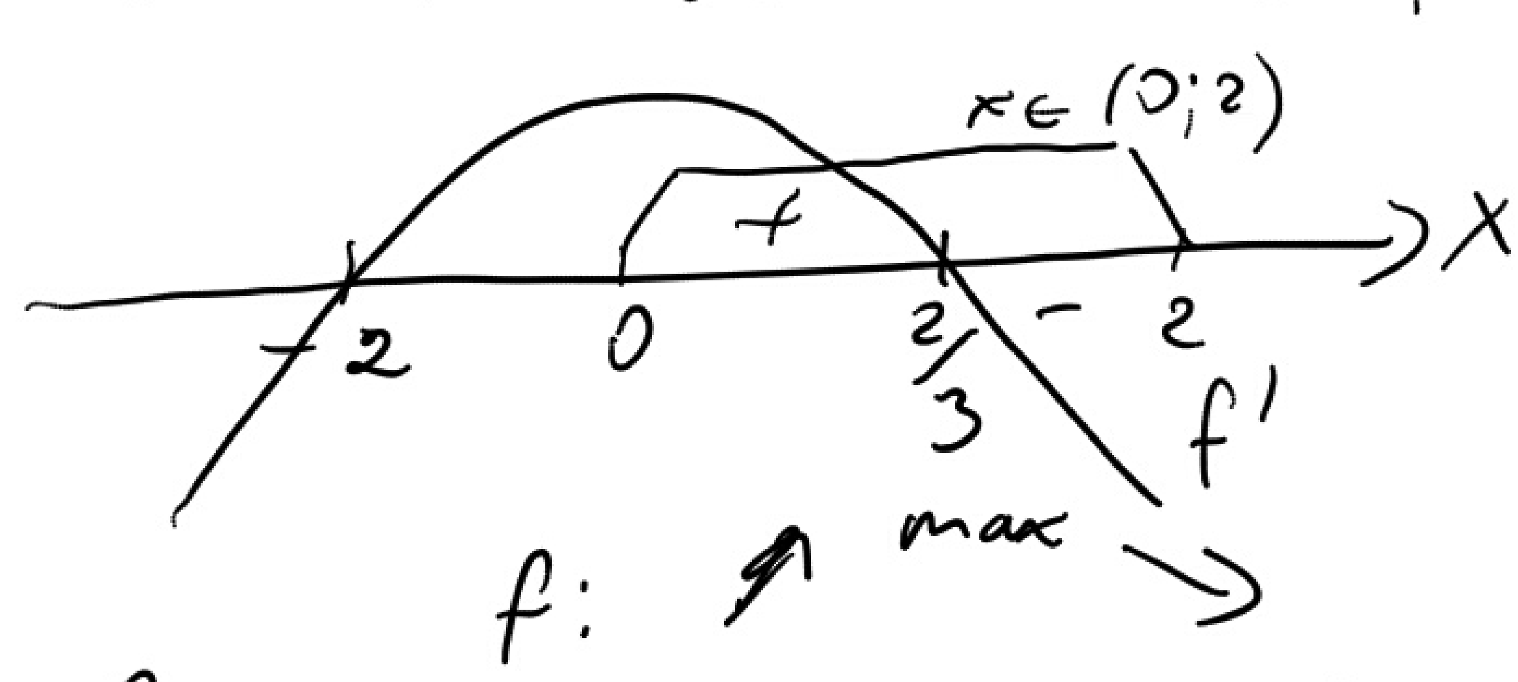
$$P_{\square} = f(x) = \frac{4+2x}{2} \cdot \left(2 - \frac{1}{2}x^2\right) = -\frac{1}{2}(x+2)^2(x-2) = -\frac{1}{2}(x^3 + 2x^2 - 4x - 8)$$

5) $f'(x) = -\frac{1}{2}(3x^2 + 4x - 4) = -\frac{3}{2}(x+2)(x - \frac{2}{3}) \wedge x \in (0;2)$

$f'(x) > 0$ dla $x \in (0; \frac{2}{3})$

$f'(x) < 0$ dla $x \in (\frac{2}{3}; 2)$ $\Leftarrow$

$f'(x) = 0$ dla $x = \frac{2}{3}$



$f(x) = \max$ dla $x = x_c = \frac{2}{3} \wedge y_c = 2 - \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 = \frac{16}{9}$

więc $C = \left(\frac{2}{3}; \frac{16}{9}\right) = \left(\frac{2}{3}; 1\frac{7}{9}\right)$

Odp: Pole trapezu w zależności od pierwszej współrzędnej wierzchołka C ma postać $f(x) = -\frac{1}{2}x^3 - x^2 + 2x + 4$ dla $x \in (0;2)$. Współrzędne wierzchołka $C = \left(\frac{2}{3}; 1\frac{7}{9}\right)$.